7 Practices of quantification from a socio-cultural perspective

Geoffrey B. Saxe

Children are engaged with mathematics in their everyday activities. Look around you – notice the children chanting numbers as they jump rope or haggling over their scores in handball. Children in urban centres in Brazil buy and sell goods (Carraher, Carraher and Schliemann 1985; Saxe 1991); toddlers in working and middle-class homes in the US play number games and sing number songs with their mothers (Saxe, Guberman and Gearhart 1987); inner-city teenage boys keep track of their statistics in league basketball play (Nasir 2002). Mathematics is interwoven in children's everyday collective activities, and yet the cognitive-developmental study of children's mathematics has often overlooked such activities as sites for analysis. The result is that treatments of development often do not capture adequately the role of children's participation in collective activities, nor the way that children themselves contribute to the mathematical norms, values and conventions that take form in collective life.

The purpose of the chapter is to present a conceptual framework for analysing the interplay between individual and collective activity in cognitive development, using mathematical cognition as an illustrative case. The framework is rooted in an assumption common to psychogenetic treatments, whether structural-developmental (Piaget 1970) or activity-theoretic (e.g. Leontiev 1981): New cognitive developments emerge as individuals create and accomplish goals in daily activities. In this chapter I present an exposition of the framework with particular attention to the 'what?,' 'how?,' and 'why?' of development.

Throughout the chapter, I limit my discussion of cognitive development to practices of quantification. Of course, quantification practices are not the only 'what?' of development, but I view quantification as a valuable arena for

I am grateful to The Spencer Foundation for support during the preparation of this chapter (no. 200100026). The data presented, the statements made, and the views expressed are solely the responsibility of the author. Various individuals contributed to this paper. Maryl Gearhart provided extensive comments and recommendations for revision. Participating members of the Fractions Research Group at UC Berkeley also contributed to the framing of the chapter. They include Britte Cheng, Rachel Coben, Jenny Garcia de Osuna, Julie McNamara, Orit Parnafes, Behnaz Shahidi and Edd Taylor. Author's email address saxe@socrates.berkeley.edu

exposition of my approach. I begin by considering practices associated with activities outside of school to highlight the cultural roots of quantification. I then go on to explain a genetic approach to exploring the 'how?' of development, illustrating with practices in school. I end with some consideration of the 'why?' of development, focusing on supports for developmental change.

The 'what?' of development: quantification practices

By quantification practices, I mean socially patterned ways in which individuals draw upon cultural forms (like number words, rulers charts, and geometrical shapes) to construct and accomplish mathematical goals in everyday activities. Children are engaged with a wide range of quantification practices both in school and out.

Consider the collective activity of Brazilian child candy sellers that I documented as sellers plied their trade (Saxe 1991). At the time of my observations, the Brazilian economy was in a period of rapid inflation. The price of a wholesale box of candy of 30, 50, or 100 units ranged between 6,000 and 20,000 cruzerios, and these prices surged at irregular intervals at each of more than 30 downtown wholesale stores. In the streets, it was common to find boys[1] selling candy to individuals at bus stops, outdoor cafes and on pavements. The candies that they sold were of various sorts (hard candy, chocolate bars, wafers) and for various prices, but always constructed as a retail price ratio of a specific number of candies for either 1000 cruzeiros (e.g. 5 for 1000) or 500 cruzeiros (e.g. 2 for 500).

The values that candy sellers computed were large, and yet many sellers were unschooled, so how did they establish their retail prices? Many of the sellers could not read the numerals on the cruzeiros notes (they knew the notes by the pictures), and they had no knowledge of school algorithms for computation. One common quantification practice was to empty the contents of a just-purchased 30, 50 or 100-unit wholesale box onto the ground, and then return candies to the box in groups of 2, 3 or 4 that corresponded to several possible price ratios, such as '2 for 500' or '3 for 1000'. Children repeated the groupings until all candy was returned (see Figure 7.1). Thus, for a box of thirty candies, if a seller returned three at a time adding 1,000 cruzeiros for each return, the sum after ten returns would total 10,000 cruzeiros. If the total turned out to be about double the value that he had paid for the box, he would decide to use the price-ratio he had just used to reconstitute the box. If not, he would re-empty the box, make an adjustment in the price ratio and repeat the process. In this way, many sellers created a retail price that afforded them an

[1] Virtually all sellers were male.

P1: FXS
0521825792c07.xml CU1669B-Demetriou July 1, 2004 2:14

Figure 7.1. A child computes what the gross price a wholesale box would yield if he sold units to customers in the street at 4 for 1000 cruzeiros.

adequate profit as well as likely sales. Widely used in the selling community, this double procedure was a convention termed 'meio-pelo-meio'.

Some features of quantification practices

The candy sellers' activities illustrate some general features common to quantification practices. These features are often neglected in theoretical and empirical treatments of the 'what?' of cognitive development.

Cultural forms in quantification practices

To price candy for sales, sellers utilize a wide array of cultural forms like candies, candy boxes, currency values and number words. When sellers return candies to the box in groups, they count candy groups using words to signify currency values (i.e. 'one thousand cruzeiros', 'two thousand cruzeiros', 'three thousand cruzeiros', . . .). The box is a form that serves as a convenient repository that represents the wholesale purchase and thus the completion of the calculation of the street price. These are just examples. The notion that individuals make use of cultural forms to accomplish problems is key to understanding quantification practices. These forms both are constitutive of problems and they are instrumental in the accomplishment of problems, and in an important sense, they are constitutive of sellers' quantifications. To date, developmental analyses of the way forms are appropriated and organized in the daily activities of children are limited (notable exceptions include Vygotsky's seminal works (Vygotsky 1978, 1987) and more recent cultural-developmental accounts (Cole 1996 and Wertsch 1985)).

Mathematical means and goals in quantification practices

Though cultural forms themselves are constitutive of quantification practices, in themselves, forms contain no intrinsic mathematical meaning. Rather, the meaning of a form emerges relative to the goals of individuals (and forms afford particular kinds of goals). For example, a seller has the goal of selling his candy to customers to accomplish the exchange he uses a particular price ration as a means of regulating the number of candies exchanged for a value of currency. In contrast, later the seller may use the price ratio as it is used in the mark-up activity as a means of accomplishing goals that involve computation of appropriate retail price. For example, after purchasing a box of fifty units and emptying the box on the ground, a seller may replace the candies of groups of five, in accord with a price ratio like '5 for Cr$1000', adding 1000 cruzeiros for each replacement until all fifty units are exhausted. In this process, he coordinates his groupings, placements, and successive additions, turning the selling ratio in a mathematical means to accomplish the goal of mark-up. Thus, in activity, individuals turn forms into means to accomplish emerging goals and in the process the same form may take on different mathematical properties.

Presuppositions in quantification practices

In the context of an everyday mathematical activity like selling candy, interlocutors tend to assume that they are doing mathematics in the same way, and they do not bother to explain each action or utterance. Yet presuppositions about the meanings of forms may not be shared; the 'same' actions and utterances may serve different functions for different individuals. For example, a younger seller may interpret an older seller's vocal count of 'one, two, three' as simply a count of groups of candies, when the older seller is in fact counting cruzeiros in an accumulation of potential sales (as in 'one thousand', 'two thousand', etc.). The older seller uses simple number words to track a progressive summation of the money gained by each sale. His counting is efficient, and his abbreviations pose no problem to himself. But what does the younger seller make of the older's quantification? The younger seller uses different presuppositions about the activity of the older and normative ways of using number words. Asymmetry in presuppositions can become a source of problems in communication if interlocutors interpret the same forms as serving different functions.

Relations between practices of quantification and social history

Quantification practices are socially and historically situated in a number of ways. First, activities like candy selling with which particular practices are associated themselves emerge and shift over historical periods. It was not always the case that selling candy was an occupation of urban street children in Brazil's Northeast; nor was it the case that there was always an urban environment. Indeed, the social organization of candy selling took form during a particular period in history and itself was situated in relation to a web of changing economic and political conditions. Further, cultural forms like number words, cruzeiro notes and even the size of a whole box take form in varied activities unrelated to candy selling and the socially recognized functions to which they were linked are each linked to historical periods. Moreover, in the hands of individuals participating in socially organized activities like candy selling the normative functions that these forms serve shift as well. For example, the meio-pelo-meio convention, for example, was reported to have had historical roots in rural life; it was viewed by many as an idiom for a particular way of conceptualizing work and profit. It has taken on a new function in the hands of sellers.

In sum, quantification practices are at the crux of the doing of mathematics in everyday collective life. They are rooted in the understandings that individuals bring to activities, and take on mathematical properties as individuals construct goals and make efforts to accomplish them. Though practices are constructed in activity, they are not the independent inventions of individuals. Rather, practices are socially and historically situated, constituted as individuals draw upon cultural forms (and the functions they afford) that themselves have complex social histories.

The 'how?' of development

How do quantification practices develop? In this section, I offer a method of analysis to explore the question. To support the exposition, I contrast my analytic approach with some features of Piaget's well-known structural-developmental investigations. Of particular concern are (1) the broad questions that frame inquiry across the two approaches and (2) the genetic methods and empirical techniques used to address core framing questions.

Piaget's structural-developmental approach

Piaget's accounts of cognitive development and his treatment of mathematical cognition hardly need introduction. His seminal analyses of cognitive structures and qualitative changes in structures over age have had a lasting influence on cognitive and developmental studies. Here, our concern is merely to note that, in his focus on cognitive structures, Piaget sidesteps an account of practices of quantification. The reason for this is not one of simple neglect. Rather, it is deeper than this, rooted in the orienting questions and units of analysis that are foundational to his analytic approach.

Piaget's starting point for inquiry is a set of questions about mathematics that are epistemological in character – what are the structural properties of mathematical knowledge (like operations and their inverses), and what is the relation between mathematics and logic (Piaget 1970)? For Piaget, the child is an 'epistemic subject', and psychological investigations are a means of empirical inquiry into cognitive structures that are posited as universal in their origins, their developmental trajectories, and in their timeless logical properties.

Piaget's method of inquiry into mathematical cognition is well represented in the *Child's Conception of Number* (Piaget 1965). Piaget begins the volume by identifying his targets of analysis – mathematical relations and concepts, including numerical one-to-one correspondences (cardinal correspondences, ordinal correspondences), quantitative invariants (conservation of discrete quantity, continuous quantities), and arithmetical operations (additive compositions). Though his framing questions are epistemological, his techniques for investigation are empirical. Through clinical interviews with children, he seeks to reveal the psychogenesis of these fundamental mathematical ideas. A core theoretical argument for which he seeks to produce support is that fundamental mathematical ideas are rooted in the development of operations and their inverses linked to a logic of classes (negations) and a logic of relations (reciprocities).

Piaget's empirical techniques involve the presentation of a wide range of carefully designed number tasks to children, exploring properties of their understandings and their logical basis through clinical interviews. For example, in one version of his well-known conservation tasks, Piaget presented children with a number of counters, asking them to produce the same number from an available set. Younger children produced solutions in which the configuration of their copy showed similarities to the model set – for example, some children aligned the endpoints of the two sets. Slightly older children established equality based upon one-to-one correspondences; however, if one set was then spread apart, they would revert back to an analysis based upon endpoints or be unsure about equality, sometimes noting co-variations in spatial extents and element separations. Still older children argued that the sets necessarily had the same number after the spatial transformation, often arguing that the greater spatial extent was compensated by the greater spatial separation, an equation of differences.

Piaget used his findings on such tasks to argue that the development of conservation understandings shifted qualitatively over age, and that such changes were related to the development of logical operations of class and order. Piaget argued that children come to equate the changes in the length of a set and element separations through a coordination of order relations (ordering spatial lengths and spatial separations) and classification (classifying the difference in length and element separations as equivalent). Through such arguments, Piaget sought to support claims about the epistemological roots of number. Thus, Piaget's argumentation borders epistemology and developmental psychology.

Piaget's focus was on universal structures of mathematical knowledge and their roots in ontogenesis. Issues of the historical and social conditions of development were hence far in the background. Indeed, Piaget argued that cognitive structures develop independently of local social and historical circumstances (Piaget 1970). Cultural issues were generally equated with factors that may affect the rate of structural development, but not much more (Piaget 1972). From this orientation, quantification practices become stripped of their social and historical properties – the forms and functions that they serve in activity.

A cultural-developmental perspective

A cultural approach begins with a different set of orienting concerns related to relations between the child and social history. The child is not only an 'epistemic subject' engaged in particular kinds of conceptual coordinations, but also an 'historical subject'. Children engage in practices of quantification in particular communities with particular social histories; they participate in collective life in particular moments of historical time. The roots of this view lie in Vygotsky's seminal writings (Vygotsky 1978, 1987), and the perspective is elaborated in current treatments (see, for example, Cole 1996; Wertsch 1991).

This analytic approach is founded on the assumption that there is a reciprocal relation between the genesis of collective activity and the genesis of individual activity.[2] On the one hand, collective activities have their genesis in the concerted work of individuals. In candy selling, the interconnected actions of sellers, clerks, customers, all working with cultural forms like currency, candy and written representations of prices, create and re-create a pattern of social organization that endures. Indeed, the collectively patterned activity of candy selling is sustained over many years, even though the particular actors change. But on the other hand, individuals within these activities are each actors, with their own beliefs, understandings and motives. Each seller constructs and accomplishes mathematical goals as he plies his trade and his goals are his own constructions.

Working within a cultural-developmental approach, I find it useful to conceptualize the 'genetic method' as requiring three different but related strands of analysis. (1) Microgenesis: how do individuals make use of forms like currency, or candy, or number words in moment-to-moment activity? On a given occasion, how do these forms come to serve particular functions as goals emerge and are accomplished? (2) Ontogenesis: how do the forms that are used and

[2] The perspective is consistent with Vygotsky's early discussion of units of analysis in the study of relations between speaking and thinking, as well as activity theoretic perspectives on activity as a nexus of relations between individual and collective activity (Leontiev 1981).

the functions that they serve shift with age and increased participation in collective activities? (3) Sociogenesis: how do new forms and functions emerge, spread and come to be valued in the quantification practices of individuals and groups? Let's consider how these three genetic concerns might frame an analysis of sellers' mark-up practices.

Microgenesis

I noted earlier that forms do not have fixed functions. Candies can be treated as food to eat, objects to count, or commodities with monetary value. None of these functions is an inherent feature of candy; rather, the functions of candy take form in activity. A *microgenetic* analysis frames questions about the process whereby individuals turn cultural forms like currency, number words and price ratios into mathematical means for accomplishing emerging goals in activities.[3] Let's consider further the way that a seller uses the price ratio form (from selling) to mark up wholesale prices.

The price ratio's original function is to mediate customer-seller exchanges. In a microgenetic analysis of the mark-up practice, we find that sellers uproot the form from its original context and use it in new ways – the price ratio becomes a mathematical means that aids the seller in planning how to sell.

In this activity, the child makes use of the price ratio to create many-to-one correspondences between multiple candies and units of 1000 cruzeiro notes, modelling repeated sales transactions. As argued earlier, though the price ratio form and its use to mediate customer-seller transactions afford the use of the ratio in mark-up practices, the generative properties of these correspondences are not contained in the ratio form or in the act of selling. Rather their generative power emerges from the child's construction of a logic of many-to-one correspondences in their moment-by-moment activity – as candies and currency values are conceptualized in relation to one another. We find evidence of the logical properties of these correspondences in sellers' sometimes repeated adjustments during mark-up. If a calculated gross price is too low after back-to-box placements (based upon the 'meio-pelo-meio' convention), a seller anticipates how to raise the value by decreasing the number of candies in his to-be-sold groups, and then repeats the back-to-box placements. If too

[3] Some researchers have made use of 'microgenesis' to refer to a methodological approach involving the intensive study of shifts in children's strategies and/or cognitive structures over short periods (see, for example, Siegler and Crowley 1991). My use of the term is more consistent with earlier treatments of the construct (Vygotsky 1986; Werner and Kaplan 1963) in which the very process of schematization of a phenomenon, perceptually or conceptually, is understood as a short-term developmental process. As conceptualized in the present discussion, microgenesis is neither a methodological approach nor a small-scale version of ontogenetic change. Rather it is a process in which forms with the cognitive functions they afford are transformed into means for accomplishing emerging goals.

high, he compensates through a logical calculation by adding a candy to his to-be-sold groupings. In these ways, the ratio becomes a mathematical means for mark-up as it is incorporated into a system of mathematical relations generated by the seller but embedded in the activity of selling candy.

An analysis of microgenesis seeks to understand the transformation of forms and functions into means and goals in activity. However, it does not address questions of the origins of these forms and functions in ontogeny. I turn next to this issue.

Ontogenesis

An *ontogenetic* analysis frames questions about shifting relations between the acquisition and use of cultural forms and mathematical functions that these forms are used to serve over age and experience. To illustrate, let's compare, for example, the practices of younger and older candy sellers. Analyses of these cohorts use of the price ratio form reveals three patterns of shifting relations. First, as sellers enter into the selling practice at any age, the price ratio is appropriated to serve an important function – a means of mediating exchanges of candy for currency with customers.

Second, with increasing age, sellers use the price ratio to serve the function of mark-up. Younger sellers do not mark-up their wholesale boxes. Rather, they either ask or are told the ratio to use by others (parents, sibs, peers, store clerks). It is only the older sellers that use the price ratio to serve in the computation of an appropriate retail price.

Third, among the older sellers who use the ratio in mark-up, we find a progressive abbreviation of the ratio form with age and experience. As sellers begin to engage in mark-up computations, they often directly model sales. For example, one approach is to pretend to sell their candy in order to compute mark up, removing candy by groups at a particular ratio sale price to compute a street price for the box. In contrast, an older seller is more likely to produce a quick and abbreviated computation in which only a trace of the price ratio is visible. For example, a seller may count the candy in a 30-unit box with single numbers in two groups of three at a time as depicted in Figure 7.2, not even removing the candy from the box. Through this procedure, the seller concludes that the box would sell for '10,000 cruzeiros', making explicit that 'ten' means '10,000 cruzeiros'.

Such form-function shifts over ontogenesis are related to individuals emerging capabilities at coordinating mathematical relations of many-to-one correspondences. For younger sellers, many-to-one correspondences are simple counts – one Cr$1000 note and a number of candies. For older sellers, many-to-one correspondences have multiplicative properties (e.g. five sales of three for Cr$1000 is equivalent to a sale of fifteen for Cr$5000). The gradual

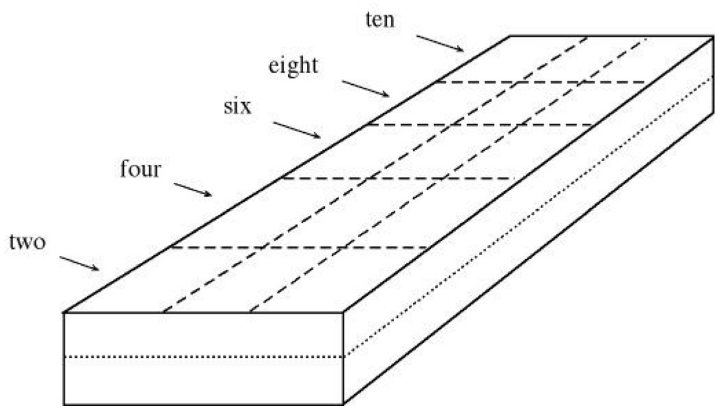

Figure 7.2. A seller's abbreviated use of the price ratio

shift from additive to multiplicative coordinations of relations over ontogenesis may well both give rise to and arise from mark-up activities. As sellers begin to engage in mark-up, they may explore the consequences of shifting the number of candies per note and exploring the effects of such alterations on the gross of potential sales.

Thus, over ontogenesis, we find a shifting relation between the forms used and the functions that they serve in quantification practices. Forms initially that may be used for a more elementary mathematical function, like the price ratio in seller–customer transactions, may gradually be used to serve new more complex mathematical functions in activities like in planning for sales in complex mark-up computations.

Socio-genesis

A socio-genetic analysis frames questions about the social origins and travel of cultural forms and their mathematical functions. Consider socio-genetic processes that occurred in approaches to mark-up with the price ratio. During the period of my study, inflationary surges led sellers to decrease the number of units sold for 1000 cruzeiros. These adjustments were often influenced by the pricing of others, leading to more general shifts for the price of candies in the streets. Similarly, during earlier periods, lower currency denominations were the valued root of the ratio – like the 500-cruzeiro note or the 100-cruzeiro note. Again, changes in the root bill value were probably initiated by some sellers, travelling through the community as new norms for selling; in turn, such shifts perhaps initiated in the mark-up by some led to shifts in selling practices as well as in the organization of mark-up computations by others.

In sum, my analytic tack is a departure from structural developmental analysis. In the Piagetian treatment, quantification practices are either analytically removed from an analysis of logical structures or bypassed by a focus on the results of clinical tasks that are often constructed without much concern for revealing the properties of cognition relative to collective life. In contrast, the focus here is on a dynamic cultural ecology of mathematics in which quantification practices are a key nexus in which we find a dynamic interplay between processes of micro-, onto-, and socio-genesis of cognitive activities.

The what?, how?, and why? Of development: a focus on practices valued in school

To explore the utility of the culturally oriented approach to developmental analysis, I turn to the practice of 'fair sharing' – dividing a quantity into equal fractional portions. I make use of the constructs sketched to provide a perspective on the what? how? and why? Of development with regard to issues related to rational number.

Fair sharing is a common pragmatic strategy outside of school and it is also a typical instructional strategy for introducing children to fractions in elementary school. As in the case of candy sellers' mark-up computations, children's mathematical concern in fair sharing out of school is practical – to resolve claims to a limited resource, say a candy bar or a brownie. Children divide the resource with concern for an equitable portion. The motive is to obtain what is fair (if not an advantage!) over other stakeholders. Children are not necessarily concerned with quantifying the portions as fractional amounts, and they may not employ fractions and fractions words at all.

However, in school, teachers treat fair sharing in new ways. Curriculum designers and teachers have appropriated the out-of-school activity to support children's developing understanding of fractions. The instructional form of the activity typically engages children in using geometrical shapes (representing a commodity like brownies, cookies or pizza) and fraction words and numeric notation to refer to parts of those shapes. The register of fraction words, written notation and geometrical shapes are the forms used in practices of quantification in elementary school, and these are the targets of our developmental analyses.

A teacher's effort to draw upon out-of-school fair sharing activities to support a lesson on equivalent fractions

Consider an episode drawn from a fourth-grade class in which a teacher makes use of a fair sharing narrative to introduce fractions. We will see that some children interpret a 'fair share' as they would out of school, and differences between the presuppositions of the teacher and the students lead to problematic

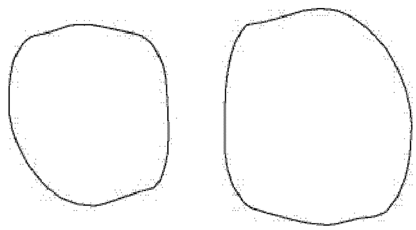

Figure 7.3. Ms Gates' drawing of the two 'huge cookies'

communications and repairs of various forms used in the activity (fraction words, geometrical shapes) and the mathematical functions that they are used to serve relative to fair sharing.The teacher's eventual recognition of the discrepancies leads to a pedagogical opportunity.

The teacher, Ms Gates, sits on a chair near the blackboard, and she tells her class about her recent trip to San Francisco. She describes her purchase of two huge cookies, and offers to share them among four students in the class.

1. TEACHER . . . I get these huge cookies [beginning to draw the first cookie on the blackboard – see Figure 7.3]. Two of them [drawing the second cookie]. And I say I am now going to choose four people. Derek, and Carrie, and Zed, and Ben [pointing to four children in her class]. And I am going to give – I'm going to share these cookies between these four people.
2. CHILD Equally?
3. TEACHER Equally! Of course. Are we democratic? Are we fair? Of course, equally!
4. CHILDREN [Background talk]
5. TEACHER Who can tell me [over the voices], wait a minute . . . [waiting for the noise to subside] how much cookie or cookies would each of those people get? [T calls on a child with his hand up] Lenny.
6. LENNY They'd each get two fourths.
7. TEACHER They'd each get two-fourths. [T now moves back to the board with chalk in hand.] So, what Lenny has done is he's said [T begins to partition the first drawn cookie with perpendicular lines through the first circle so that the drawn cookie is partitioned into quadrants. She's only partially finished when another child calls out]
8. CHILD FROM CLASS [Gary?] or one half
9. TEACHER Ok. It's going to help; it's going to really help if people don't call out.
10. ANOTHER CHILD [[gary?] two fourths or one half
11. MORE CHILDREN [muffled task-related talk]
12. TEACHER [continues partitioning of the first circle into quadrants] I think this is why – tell me Lenny now am I wrong? You were saying that we'll get one cookie and we'll divide it between four people. So, each of them will get one of the quarters. And we divide the second cookie among the four people, so if it was Joey getting them, he'd get a piece from each.
13. CHILDREN [lots of task-related talk]
14. TEACHER And so Lenny said they'd get two quarters.

15. BACKGROUND CHILD TALK [lots more talk]
16. TEACHER [continues]. . . right, ok . . . [Teacher pauses] Lenny told me that it would be 'two-fourths'. He also then added something to that – what did you tell me Lenny?
17. LENNY [getting up and going to the board] The reason why I did two fourths was because one cookie is bigger than the other [pointing to each cookie].
18. TEACHER Oh. . . . So . . . [looking momentarily confused] they'd get an equal quarter from each one. But you know what they'd get an equal quarter from each one. I'd have to make sure. So, do you hear what Lenny's saying? I just didn't divide them into halves like this, because you know the people getting half from here wouldn't get as much as those getting half from here. So I've given them 1/4 of this one and a fourth of this one, and together they'd have a whole cookie. And Glen's point was you didn't need to do them that way. You could just divide them into half. But you'd have to make absolutely sure the cookies were the same size . . .

In this short episode, the teacher poses to her class a fair share problem with some similarities to many children's everyday activities – if two cookies were shared among four people, how much would each person receive? But in the instructional context, the teacher's and the children's interpretations clash, and communications are repaired. Let me now discuss how the cultural–genetic framework frames the analysis of developmental processes in the quantification practices captured in this episode.

Microgenesis and the fair sharing episode

During the whole class discussion, Ms Gates and Lenny are drawing upon the same cultural forms – particular fraction words and the drawn circles – but they are using them in different ways. For each of the participants, these forms become representations of quantities; the fraction word, 'two fourths' refers to particular mathematical relations between parts and wholes. However, Ms. Gates' 'two fourths' and Lenny's 'two fourths' carry different meanings.

In her representation of the fair share, 'two fourths', Ms Gates is engaging in what is a normative practice in schools, one in which graphical representations are treated as idealized quantities. She assumes that two crudely drawn circles will be taken as two circular cookies of equivalent size. This idealization appears to be grounded in her intent to teach a lesson on equivalent fractions through the example that 'two fourths' (as supported by the partitioning of circles in Figure 7.4a) is equal to 'one half' (as supported by the partitioning of circles in Figure 7.4b).

The microgenetic origins of Lenny's construction emerge from different presuppositions. Lenny assumes that the cookies are different sizes (line 19), an interpretation that is faithful to the blackboard drawing (Figure 7.4c). Thus, for Lenny, the appropriate solution is one-fourth from each cookie, not one-half from either. He interprets the goal of the task in terms of the pragmatic

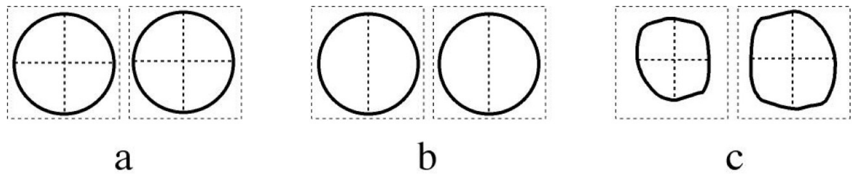

Figure 7.4. Three partitionings of Ms Gates' cookies.

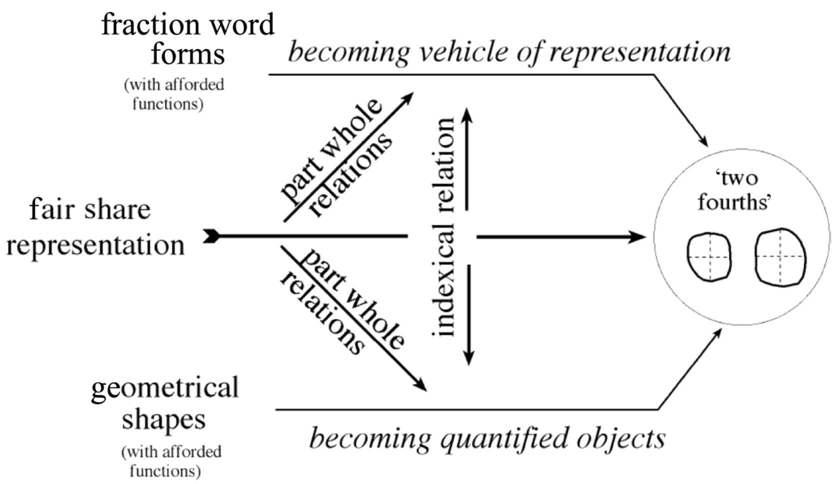

Figure 7.5. A microgenetic representation of Lenny's 'two-fourths'.

concerns of out-of-school sharing practices – he wants to be certain that the fair shares are equal.

The microgenetic processes in Lenny's and Ms Gates' activity reveal some formal similarities, even though the genetic origins of their microgenetic trajectories differ. In each case, we find a process that entails related strands of activity as depicted in Figure 7.5: treating fraction word forms as representations of quantities; partitioning shapes such that they become representations of quantities; and creating a relationship between word forms and shapes such that one serves to index the other.[4] Let's focus just on the microgenesis of Lenny's representation.

As depicted in the lower line, geometrical shapes become quantified objects as they are partitioned as fair shares of a whole. Lenny takes the two shapes as different sizes, each a different set of part–whole relations. As depicted in the

[4] This account shares some similarities with that which Werner and Kaplan (1984) present in their volume, *Symbol formation*.

P1: FXS
0521825792c07.xml CU1669B-Demetriou July 1, 2004 2:14

higher strand, fraction words become treated as word forms that can carry the meanings of part–whole relations. For Lenny, the 'two' in 'two "fourths"' represents a cardinal value – the number of non-equivalent fourths. The meaning of 'fourths' is based on his interpretation of the value of each cookie as drawn. The dynamics of these processes are interrelated as depicted in the middle strand. The functions of word forms and shapes are constructed in relation to one another, such that the vehicle – in this case a cardinal word and fraction word – comes to index the object, in this case, a part of each cookie.

In Ms Gates' construction, there is a similar creation of quantified objects, representational vehicles and indexical correspondences. However, the process is rooted in different presuppositions regarding units, representations and the meaning of partitioning operations.

Sociogenesis and the fair sharing episode

Socio-genetic analyses focus on the emergence and spread of quantification practices. The use of geometric shapes for teaching fair share distributions is not an instructional innovation of Ms Gates' own design. The approach is common in curriculum units, and Ms Gates borrows and adapts it. Lenny brings his version of the fair sharing practice to the lesson. In her efforts to repair the communicative problem, Ms Gates tries to explain that there are two ways of taking the shapes as mathematical objects – idealized representations of quantities and their actual sizes. As this curriculum unit unfolds over the next few weeks, students will incorporate these two perspectives as they make sense of fair share solutions.

In sum, though practices are the construction of individuals, they certainly are not independent inventions. In the dynamics of collective activities, individuals are drawing upon forms with social histories and upon others' uses of such forms in the course of activity. It is in the context of use that innovations may emerge and spread as individuals interact with one another, making use of one another's constructions.

Ontogenesis and the fair sharing episode

The students in Ms Gates' classroom come to the fair share activity with prior experience with number and fraction words, geometrical shapes and activities of partitioning. Indeed, their interpretations of fair sharing have roots in much earlier periods of cognitive development when they used word forms, geometrical shapes and actions like splitting to serve functions that had little to do with the representations of fractions. In ontogenetic analyses, the concern is to understand these roots and trajectories of developmental change, with particular regard for the shifting relations between forms and functions in quantification practices. Let's consider some early activities that are arguably the ontogenetic roots of the quantification practices that Ms Gates is supporting in her classroom.

In early childhood, children engage in a wide variety of number activities. They use cardinal and ordinal number word forms in activities that include recognizing numerals on playing cards with parents, singing counting songs, telling their age, and racing with their friends (Saxe, Guberman and Gearhart 1987). Children also learn to name and differentiate geometrical shapes, including triangles, circles, squares and ovals, from picture books, educational activities involving plastic shapes and educational software. Finally, children engage in activities in which they split objects, as in out-of-school fair sharing activities. These varied forms are all ingredients of the practices that Ms Gates wants to support in her classroom, but in early development these forms serve very different functions than the representation of fractions.

To support an analysis of the ontogenesis of children's representational practices involving fractions, I am currently investigating how children come to use fraction words and geometric shapes to represent fractional quantities. My graduate students and I are interviewing third, fourth and fifth graders. We are finding that some children use whole number forms to represent their emerging understandings of part–whole relations (depicted in a drawing), while other children use fraction words to represent whole number conceptions. Consider how younger children responded to item 6a. Some children represented the fractional part as 'one out of four' (instead of the canonical form, 'one fourth'); other children said 'one third', using the canonical word form to represent a whole number counts of the shaded pieces. Older students tended to use the canonical form in response to all items, and they were typically correct on item 6a. However, many students did not well differentiate the function of fraction words to represent discrete and continuous quantities, referring to 6b as 'one fifth', suggesting that while the used a canonical form to serve the function of representing a relation between parts and whole, they were not differentiating area, a continuous quantity, from a discrete quantity.

'Whys?': sources of change

Children construct new quantification practices through participation in collective activities like fair sharing and candy selling. Is it possible to identify the properties of such activities that support developmental processes of the sort that Ms Gates values in her classroom? I end the exposition by sketching a program of research that pursued this question.

The research focused on relationships between instructional practices and children's developing understandings of fractions. We tracked upper elementary children in twenty-one classrooms over the course of their instructional

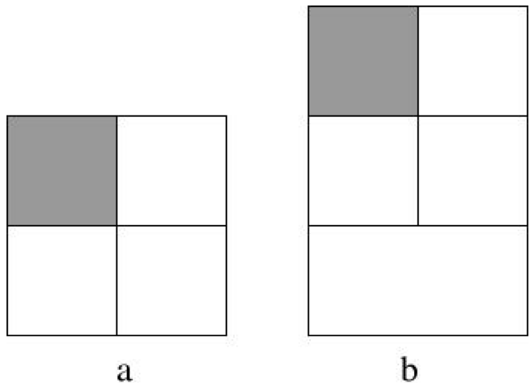

Figure 7.6. Grayed parts of areas used in a cross-sectional study with 3rd, 4th, and 5th graders 'Whys?': sources of change.

units (over 300 students participated). In each classroom, we followed two types of students from pre- to post-test – students who began instruction either with or without incipient understandings of fractions based on pre-test performance (Saxe, Gearhart and Seltzer 1999). We also rated the quality of fractions instruction based on observations of key lessons. With the rating system, we did not simply focus on the teachers' instructional moves. Rather, the concern was to capture the classroom as a collective, one in which student participation as much as teacher participation contributed to judgments about the quality of students' opportunities to learn. Gearhart, Saxe, Seltzer, Schlackman, Ching, Nasir, Fall, Bennett, Rhine and Sloan (1999) contains a complete description of this rating system as well as findings that bear on its validity as a measure of opportunities to learn in classrooms.

We expected that the two types of students would tend to generate different kinds of mathematical goals in their instructional activities, leading to differences in changes from pretest to posttest performances depending on the quality of instruction. Children without incipient understandings would tend to interpret instructional tasks in terms of whole numbers, not part–whole and multiplicative relations, unless instructional practices guided them to reconceptualize quantitative relations. In contrast, children with an incipient understanding of part-whole coordinations on elementary fractions tasks would be much more likely to progress during the unit even if instructional practices provided less support.

The findings are reported in Saxe, Gearhart and Seltzer (1999), and I sketch one strand of analysis here. Figure 7.7 is a plot of mean post-test performances[5]

[5] Post-test scores are adjusted by pre-test scores and language background of students in the classrooms.

P1: FXS
0521825792c07.xml CU1669B-Demetriou July 1, 2004 2:14

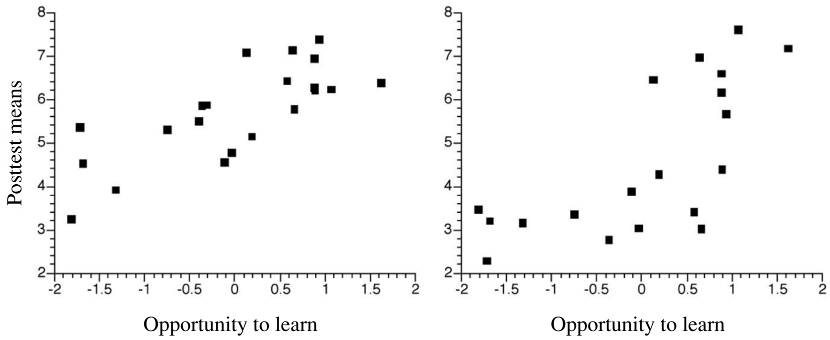

Figure 7.7. Adjusted post-test means for children with and without incipient knowledge in 21 classrooms as a function of opportunity to learn.

for each of the twenty-one classrooms as a function of the opportunity to learn ratings. The findings are consistent with our conjectures about the interaction of students' incipient knowledge and opportunity to learn. For the students with incipient understandings, the relationship between performance on post-test and opportunity to learn was linear; the greater their opportunity to learn fractions in the classroom, the more students progressed. For the students without incipient understandings, the relationship between performance on post-test and opportunity to learn was not linear. There was little gain for students whose classrooms were rated as providing limited opportunity to learn. However, there was a precipitous increase in student gain in classrooms rated more highly. Thus, when instruction is geared toward building upon what children understand, it is more likely that students – even with limited initial understandings – will make marked gains.

To deepen our understandings of these results, graduate students and I are analysing videotapes of fair share lessons collected in two contrasting classrooms to understand the interplay between micro-, onto-, and socio-genetic processes. One videotape was collected in a classroom that ranked highest on our scale of opportunity to learn; in this classroom there was no post-test gap between students who began instruction with and without incipient understandings, and the post-test means for both groups were high. The other is from a classroom that was ranked just below the median level on the opportunity to learn scale; in this classroom, the post-test scores of the group without incipient understandings were much lower than those of the incipient group.

The differences in classroom practices rated with different opportunities to learn revealed marked differences in the way students' efforts to structure and accomplish goals involving fractions were supported. In the classroom rated with greater opportunities to learn, the teacher built her instructional approach to the fair share problem on students' mathematical thinking, supporting their emerging mathematical goals as they worked to conceptualize and accomplish problems. For example, in this class, the teacher would pose a challenging problem that built on prior problems, and asked the students to solve it, encouraging them to enlist the help of a partner if needed. As she roved through the classroom, she observed students, sometimes asking them to explain their reasoning. As she queried and engaged a student in a brief conversation, she encouraged other students to comment and extend the dialogue. Students often listened and sometimes appropriated what they saw or heard into their own solutions. In the final discussion, the teacher asked students to share their strategies, and engaged the whole class in commentary on it. Her questions and comments generally followed students' line of reasoning, encouraging students to formalize their thinking and their representations. For example, she asked students to represent their work using both drawings and numeric notation, and to explain the relationships between the two.

In the classroom ranked below the median, the teacher tended to model how to partition fair shares, and she did not adopt a systematic approach to either graphic or numeric notation. In her opening lesson and in her interventions with individual students, her drawings tended to be unpredictable in size and shape, and her references to shapes and their parts were often a mix of canonical expressions (like 'one fourth') and whole number expressions, like 'a piece', or 'one piece' without well framing her intended meanings. Sometimes fractional parts were labelled with whole numbers and sometimes they were identified as fractions, but the relations between these different kinds of reference would be confusing to a child who does not readily conceptualize parts of areas in terms of fractions. When this teacher roved and when she led the final whole class discussion, she tended to correct students. She was not often engaged in understanding the students' approaches to the problem, or building the discourse on student understanding.

Concluding remarks

I end this sketch of a cultural developmental framework by briefly situating it in relation to some of the neo-Piagetian approaches to cognitive development.

I noted earlier that Piaget's major emphasis in his empirical and analytic work was to offer arguments about the origins and properties of universal cognitive structures and their developmental transformations. Piaget did not pursue analyses of individual differences nor processes whereby social factors might alter structural-developmental processes. These were largely concerns for psychology, and his occasional claims about such matters were peripheral to his seminal contributions to a genetic epistemology. In this regard, Piaget regarded the individual as an epistemic, not a psychological subject (Piaget 1970).

Neo-Piagetian frameworks have shifted methods and analytic techniques, treating the individual not only as a vehicle for epistemological inquiry, but also as a psychological subject. Methods are varied. They include cross-sectional and short-term longitudinal studies on batteries of cognitive tasks including variants of tasks use by Piaget like seriations and conservations (e.g. Case 1985; Pascual-Leone 1970). They also include training studies in which efforts are made to support children's improvements with particular regard for how learning may emerge in concert on subsets of assessment tasks (e.g. Case, Okamoto, Henderson and McKeough 1993; Demetriou, Christou, Spanoudis and Platsidou 2002). In addition, they include cross-national studies that make an effort to understand what are universal and specific developments in sets of cognitive abilities (e.g., Case, Okamoto, Griffin, McKeough, Bleiker, Henderson and Stephenson 1996; Demetriou et al. 2002). The psychological models that have given rise to and flowed from such studies have sought to explain consistencies and inconsistencies in cross-task performances over age in terms of domain specific knowledge, like number or space, and domain general constraints that regulate domain specific constructions, such as constructs like working memory (e.g. Case 1985; Pascual-Leone 1970), processing efficiency (Demetriou, Christou, Spanoudis and Platsidou 2002), and webs of semantic connections within and across domains of knowledge (Case, Okamoto, Henderson and McKeough 1993).

The neo-Piagetian arguments for domain specific cognitive structures are in accord with some current modularist positions (Butterworth 1999; Dehaene 1997) that attribute domain-specific processing mechanisms for particular kinds of knowledge (e.g. number). However, unlike these modularist accounts that tend to assume hard-wired numerical knowledge, at least some neo-Piagetian accounts posit and find support for a key role of individuals' constructive activity in the structural transformations of domain specific knowledge over development (e.g. Karmiloff-Smith 1995).

In some respects, the framework sketched in this chapter travels on some similar ground with neo-Piagetian models. In focusing on emerging mathematical goals in everyday activities, I am concerned with domain-specific construction of logico-mathematical structures and their developmental transformation. Thus, in this chapter I delineated some of the features of structural-developmental progressions in sellers' coordination of many-to-one correspondences in their mark-up activities and in children's coordination of part–whole relations in fair sharing. Further, like Piagetian and neo-Piagetian formulations, the model I am proposing of (ontogenetic) development coordinated logico-mathematical operations is epigenetic. New structural coordinations are born from prior ones. Some supportive evidence for this thesis came from sellers' appropriation of their use of many-to-one correspondences in sales to their use of these correspondences in mark-up and the gradual working out of new mark-up strategies that involved more complex coordinations of correspondence relations. Finally, the studies on fractions as a function of different classroom practices are consistent with neo-Piagetian models that posit limits that enable and constrain children's construction of new mathematical understandings under educational interventions.

However, the present framework departs substantially from neo-Piagetian psychological models in important ways. From the perspective that I have elaborated, individuals are not only psychological subjects as the neo-Piagetian models have emphasized, but also historical subjects. Indeed, whether we are discussing candy sellers in the streets of north-east Brazil or children participating in classroom communities, these children are actors enmeshed in webs of social organizations that sustain valued forms of collective representations with socially recognized functions. Such social organizations are themselves collective constructions that have been elaborated over the varied social histories of communities. Further, individuals play a constitutive role in both reproducing and producing alterations in these historical constructions through their participatory activities (Modell 1996; Saxe 1999; Saxe and Esmonde in press). In an autocatalytic process, the production, reproduction, and alteration of these forms create emerging environments that themselves become interwoven with the properties of individuals' own developmental trajectories in non-trivial ways. Thus an analysis of the individual must entail an analysis of social life, and vice versa. It is not that the psychologically oriented models do not attend to social factors. Indeed, in some impressive studies, authors like Case et al. (1996) and Demetriou, Kui, Spanoudis, Christou, Kyriakides and Platsidou (2003) have pointed to social and cultural variables that predict differences in the rate of particular developments cognitive structures or structural-like processes. The problem is that, from a historical perspective, in these models and methods of study, the constructive activities of individuals as they emerge in social life remain all but invisible.

Taking the individual as an historical subject complicates but also enriches epistemological and psychological analyses of cognitive development. It requires the elaboration of new analytic units to support empirical inquiry, ones that open up opportunities for analysing the interplay between historical and developmental processes in the micro-, socio- and ontogenetic construction of knowledge. My sketch of the work on quantification practices is one emerging effort to re-situate the analysis of cognition in a perspective that coordinates not only epistemic and psychological concerns, but also the role of the individual as an actor, participating in, drawing from, and contributing to continuities and discontinuities in forms and functions of knowledge not only in their own developments but in the social histories of communities.

REFERENCES

Butterworth, B. (1999). *What counts: how every brain is hardwired for math*. New York: The Free Press.

Carraher, T. N., Carraher, D. W. and Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology 3*, 21–9.

Case, R. (1985). *Intellectual development: birth to adulthood*. New York: Academic Press.

(1992). *The mind's staircase: exploring the conceptual underpinnings of children's thought and knowledge*. Hillsdale, NJ: Erlbaum.

Case, R., Okamoto, Y., Henderson, B. and McKeough, A. (1993). Analyzing diversity in developmental pathways: Methods and concepts. In R. Case and W. Edelstein (eds.) The new structuralism in cognitive development: theory and research on individual pathways. (pp. 71–100). Contributions to Human Development, vol. 23. Basel: Karger.

Case, R., Okamoto, Y., Griffin, S., McKeough, A., Bleiker, C., Henderson, B. and Stephenson, K. M. (1996). The role of central conceptual structures in the development of children's thought. *Monographs of the Society for Research in Child Development, 61* (1–2), serial no. 246.

Cole, M. (1996). *Cultural psychology: a once and future discipline*. Cambridge, MA: Harvard University Press.

Dasen, P. R. and de Ribaupierre, A. (1988). Neo-Piagetian theories: cross-cultural and differential perspectives. In A. Demetriou (ed.) *The neo-Piagetian theories of cognitive development: toward an integration*. North Holland: Elsevier.

Dehaene, S. (1997). *The number sense: how the mind creates mathematics*. New York: Oxford.

Demetriou, A., Christou, C., Spanoudis, G. and Platsidou, M. (2002). The development of mental processing: efficiency, working memory, and thinking. *Monographs of the Society for Research in child Development, 67*(1), serial no. 263.

Demetriou, A., Kui, Z. X., Spanoudis, G., Christou, C., Kyriakides, L. and Platsidou, M. (2003). The architecture and development of mental processing: Greek, Chinese, or universal? Unpublished manuscript. University of Cyprus.

Gearhart, M., Saxe, G. B., Seltzer, M., Schlackman, J. Ching, C., Nasir, N., Fall, R. Bennett, T. Rhine, S. and Sloan, T. F. (1999). Opportunities to learn fractions in elementary mathematics classrooms. *Journal for Research in Mathematics Education 30*(3): 286–315.

Karmiloff-Smith, A. (1995). *Beyond modularity: a developmental perspective on cognitive science*. Cambridge, MA: MIT Press.

Leontiev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (ed.) *The concept of activity in Soviet psychology* (pp. 37–71). NY: Sharpe.

Modell, J. (1996). The uneasy engagement of ethnography and human development. In R. Jessor, A. Colby and R. A. Shweder (eds.) *Ethnography and human development: context and meaning in social inquiry* (pp. 479–504). Chicago: University of Chicago Press.

Nasir, N. S. (2002). Identity, goals, and learning: mathematics in cultural practice. *Mathematical Thinking and Learning, 4*(2 and 3), 213–47.

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologica, 32*, 301–45.

(1988). Organismic processes for neo-Piagetian theories: a dialectical causal account of cognitive development. In A. Demetriou (ed.) *The neo-Piagetian theories of cognitive development: towards an integration* (pp. 25–64). Amsterdam: North-Holland.

Piaget, J. (1965). *The child's conception of number*. New York, Norton.

(1970). Piaget's theory. In P. H. Mussen (ed.) *Carmichael's manual of child psychology* (pp. 703–31). NY: Wiley.

(1972). Intellectual evolution from adolescence to adulthood. *Human Development*, 15(1), 1–12

Saxe, G. B. (1991). *Culture and cognitive development: studies in mathematical understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

(1999). Cognition, development, and cultural practices. In E. Turiel (ed.), *Culture and development: new directions in child psychology* (pp. 19–35). San Francisco: Jossey-Bass.

Saxe, G. B., Guberman, S. R. and Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development, 52*(2), serial no. 216.

Saxe, G. B. and Esmonde, I. (in press). Making change in Oksapmin tradestores: studies in shifting practices of quantification under conditions of rapid social change. *South Pacific Journal of Psychology*.

Saxe, G. B., Gearhart, M. and Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17*(1), 1–24.

Saxe, G. B. and Guberman, S. R. (1998). Studying mathematics learning in collective activity. *Learning and Instruction, 8*(6), 489–501

Siegler, R. S. and Crowley, K. (1991). The microgenetic method: a direct means for studying cognitive development. *American Psychologist, 46*, 606–20.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge, MA: Harvard Univeristy Press.

(1986). Thought and Language, Cambridge, MA: MIT Press.

(1987). *Thinking and speech*. New York: Plenum.

Werner, H. and Kaplan, B. (1963). *Symbol formation: An organismic-developmental approach to language and the expression of thought*. New York: Wiley.

(1984). *Symbol formation: an organismic-developmental approach to the psychology of language*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. Cambridge, MA: Harvard University Press.

(1991). *Voices of the mind*. Cambridge, MA: Harvard University Press.