

## Studying Cognition through Time in a Classroom Community: The Interplay between “Everyday” and “Scientific Concepts”

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### Key Words

Classroom practices · Embodied cognition · Everyday concepts · Longitudinal analysis · Mathematical cognition · Mathematics education · Number line · Scientific concepts

### Abstract

This paper presents an analytic approach for understanding the interplay through time between “scientific” and “everyday concepts” in a mathematics classroom community. To illustrate the approach, we focus on an elementary classroom implementing an integers and fractions lesson sequence that makes use of the number line as a principal representational context. In our analysis of the community’s emerging collective practices (recurring structures of joint activity), we trace the interplay between children’s sensorimotor actions (displacing, counting, and splitting) and the mathematical definitions supported in the classroom, like definitions of unit interval or equivalent fractions. In our illustrative analysis, we find that the teacher orchestrated collective practices to support the use of actions to make sense of the formal definitions, and the use of definitions to regulate actions. Though we illustrate the analytic approach for a particular classroom community, the approach illuminates teaching-learning dynamics that transcend any particular classroom or subject matter domain.

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Formal education presents teachers and students with a conundrum. Teachers are tasked with guiding and supporting students’ efforts to learn knowledge that has developed over the history of disciplines. However, the formalized character of this

disciplinary knowledge means that it is removed from the local, situated knowledge that children construct in their daily lives. How might teachers support children in reconciling this problematic divide? How might students create connections between local knowledge they generate in their everyday, out-of-school lives and what may appear as alien ideas presented in school?

In his sociohistorical treatment of cognitive development, Vygotsky [1986] cast the conundrum as a dialectic between “scientific concepts” and “everyday” (or “spontaneous”) concepts.<sup>1</sup> Vygotsky argued that these two kinds of concepts develop in different directions. Everyday concepts, which have roots in children’s reactions to local situations, develop from the “bottom up,” towards increasing levels of adequacy and generality. Scientific concepts, introduced through explicit instruction and initially understood at a shallow level, develop from the “top down” as children enrich and transform them through their conscious application to local situations. In this process, Vygotsky argued that there is a potential for an interplay between the two strands of development. Scientific concepts can provide avenues for the development of everyday concepts such that everyday concepts can develop in the direction of historically elaborated systems of generalization (e.g., algebra; Newtonian mechanics). At the same time, everyday concepts can provide avenues for making sense of developing scientific concepts so that scientific concepts become explored and fleshed out in relation to the material conditions of daily life. Of course, such a productive interplay may not be supported in instruction. Thus, students may not acquire scientific concepts at all or merely acquire what Alfred North Whitehead referred to as *inert knowledge*: “ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations” [Whitehead, 1959, p. 193]. Although scholars have called attention to the import of relations between local and disciplinary knowledge in instruction and development [e.g., Davydov, 1990; Gutiérrez & Rogoff, 2003; Hershkowitz, Schwarz, & Dreyfus, 2001; Karpov, 2003], the interplay between top-down and bottom-up developmental processes over the course of lessons has received limited systematic analysis and empirical study [exceptions include Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014; Yoshida, 2004]. Such a longitudinal analysis has the potential to further illuminate a critical nexus of cognitive-developmental and sociocultural processes; it also has the promise of informing our understanding of a core challenge of formal education.

In this article, we build upon Vygotsky’s insights about the interplay between top-down and bottom-up developmental processes. Our goal is to understand what it might look like for a productive interplay to be supported in classroom instruction as lesson topics shift over time. To leverage our analytic efforts, we focus on a fifth-grade case study classroom in which a teacher is implementing a curriculum called *Learning Mathematics through Representations* (LMR) [Gearhart & Saxe, 2014; Saxe, de Kirby, Le, Sitabkhan, & Kang, 2014; Saxe, Diakow, & Gearhart, 2013a]. The LMR curriculum is a 19-lesson sequence that provides a fruitful context for study, a context in which a potential for an interplay stands out in remarkably clear relief. The cur-

<sup>1</sup> Vygotsky originally referred to “spontaneous concepts” as “everyday concepts,” with a shift in reference occurring in response to Piaget’s usage of the term, “spontaneous concepts” [Temina-Kingsolver, 2008] [see also: Hedegaard, 1990, 2007; Moll, 1990; Nelson, 1995; Panofsky, John-Steiner, & Blackwell, 1990; Wells, 1994; Wertsch, 1991].

riculum introduces a register of 17 mathematical definitions related to core ideas about integers and fractions on the number line.<sup>2</sup> From a Vygotskian perspective, these number line definitions constitute scientific concepts. The lesson sequence also affords a treatment of everyday concepts – children’s everyday sensorimotor actions that develop in daily activities [Piaget, 1970b]. These actions can be used as a generative basis for approaching number line problems and to make sense of number line definitions. Further, the length of the lesson sequence allows for a longitudinal analysis of the collective practices that emerge in the classroom – that is, recurring forms of activity in which norms, values, and social positions are constituted and re-constituted over time [see Saxe, 2012]. Such collective practices, as we will demonstrate, provide us with an important nexus for analyzing the interplay between scientific and everyday concepts through time.

We organize our treatment in several sections. In the first, we describe properties of the LMR register of number line definitions (scientific concepts), and in the second, we present a corresponding analysis of relevant everyday concepts. In our treatment, these everyday concepts are sensorimotor actions and their coordinations – specifically, counting, displacing, and splitting actions. We show that students may productively extend these everyday actions to conceptualize and solve number line problems. The third section of the article presents our framework for understanding the interplay between definitions and actions through time. In this section, we elaborate the argument that actions provide children with the initial basis for interpreting the line as an object with mathematical properties. Further, we show how these actions also afford means of making sense of number line definitions. We then introduce a longitudinal framework that focuses on shifting relations through time between actions and definitions in problem solving activity. In the fourth section, we use the framework to analyze six episodes extracted from our longitudinal video record of our target classroom; of particular interest are collective practices that emerge in the classroom community and the way these practices provide contexts for a productive interplay between actions and definitions. In a fifth and concluding section, we consider the utility of the framework for studying classrooms that extend beyond our illustrative case and to other knowledge domains.

### **The LMR Curriculum Unit and Its Register of Mathematical Definitions<sup>3</sup>**

The LMR curriculum consists of 19 lessons that move from core ideas in integers (lessons 1–9) to fractions (lessons 10–19). Over the course of the lessons, the curriculum introduces 17 integers and fractions number line definitions. As a network of

<sup>2</sup> We use the expression “mathematical register” in ways that share features with Halliday’s original use of the term [Halliday, 1978]. Similar to Halliday, we use the term “mathematical register” to refer to the word forms that come to serve specialized functions in communication about mathematical ideas [for further discussions of Halliday’s “mathematical register” in classroom practices see also Forman, McCormick, & Donato, 1997; Schleppegrell, 2007]. At the same time, we also use the term to communicate the network of “scientific concepts” (or formal definitions) that the LMR curriculum supports.

<sup>3</sup> From this point in the article, in most instances, we will omit the term “principles” when we refer to “definitions and principles.”

Name	Definition	Example
Order (Lesson 1)	Numbers increase in value from left to right. Numbers decrease from right to left.	
0 is a number (Lesson 1)	0 is a number, so it has a place on the number line.	
Interval (Lesson 2)	The distance between any two numbers on the number line.	
Unit interval (Lesson 2)	A unit interval is the distance from 0 to 1 or any distance of 1.	
Multunit interval (Lesson 3)	A multiple of a unit interval.	
Every number has a place (Lesson 3)	Every number has a place on the line, but not all need to be shown.	
Symmetry (Lesson 6)	For every positive number, there is a negative number that is the same distance from 0.	
Absolute value (Lesson 6)	The distance of a number from 0.	

Fig. 1. Number line definitions: integers.

scientific concepts, the definitions provide a general interpretation of integers and fractions on the number line appropriate for the upper elementary grades.<sup>4</sup>

The definitions for the integers lessons are contained in figure 1. As indicated in the figure, early in the integers lessons the curriculum introduces definitions basic to the number line, like *order* (lesson 1) and *zero is a number* (lesson 1). Over the course of the integers lessons, key ideas of *interval* (lesson 2), *unit interval* (lesson 2) and *multunit interval* (lesson 3) are defined. Entailments of these ideas are also defined

<sup>4</sup> A more complete description of the curriculum as well as the design research that informed its development is contained in Saxe, de Kirby, Le, Sitabkhan, and Kang [2014]. The curriculum materials are downloadable at <http://www.lmr.berkeley.edu/>. Useful reviews of curricular approaches to linear measurement are contained in Smith, Males, Dietiker, Lee, and Mosier [2013], and a useful review of the development of measurement understandings in children is contained in Lehrer [2003].

later in the lesson sequence – *symmetry* (lesson 6) and *absolute value* (lesson 6). In addition, number line “principles” are included in the definition list, such as *every number has a place* (lesson 3).

The definitions for the fractions lessons (fig. 2) build upon the integers definitions. Early in the fractions lessons *subunit* is introduced (lesson 10), and subsequently, *subunit* is used as a basis to define *numerator* and *denominator* on the number line (lesson 11). *Mixed number* follows as a combination of integers and fractions definitions (lesson 13). Multiplicative relations intrinsic to the domain of fractions/rational number are made explicit in the later fractions lessons with the introduction of the *equivalent fraction* definition (lesson 15). As in the integers lessons, number line principles are included in the definition list. An example is *benchmarks*: “You can tell about how big a fraction is by comparing the numerator and the denominator” (lesson 18). As a set, the definitions and principles functionally integrate the domains of integers and fractions in a common framework.

As the number line is central to the LMR curriculum, we are able to build on and contribute to literatures in the learning sciences on children’s developing understandings of the number line [Booth & Siegler, 2008; Earnest, 2012; Petitto, 1990; Saxe, Earnest, Sitabkhan, Haldar, Lewis, & Zheng, 2010; Siegler & Opfer, 2003] as well as linear representations [Barrett, Sarama, Clements, Cullen, McCool, Witkowski-Rumsey, & Klanderma, 2012; Lehrer, 2003; Núñez, 2011; Piaget & Inhelder, 1956; Piaget, Inhelder, & Szeminska, 1960; Saxe, Shaughnessy, Gearhart, & Haldar, 2013b]. As a whole, these literatures reveal that young children show capabilities to order numbers along a linear dimension; however, only later do children show capabilities to generate uniform metric properties in their ordering of numbers. In our longitudinal treatment, we focus on the import of the dynamic interplay between top-down and bottom-up processes in children’s developing metric understanding of the number line.

### **Everyday Actions with Logico-Mathematical Properties: Resources for Constructing Solutions to Number Line Problems and Making Sense of Mathematical Definitions**

Various scholars have argued that schemes for sensorimotor actions provide a core resource for children’s developing understanding of logico-mathematical ideas, whether from ontogenetic [Langer, 1980, 1986; Núñez & Marghetis, in press; Piaget, 1963, 1970a; Steffe, von Glasersfeld, Richards, & Cobb, 1983] or microgenetic [Schmid-Schönbein & Thiel, 2010; Thiel, 2012] perspectives.<sup>5</sup> Piaget [1970a, pp. 16–17] provides a clear illustration in his reminiscence of a child’s construction of the commutative property of counting. In this case, the child’s construction of new mathematical knowledge is dependent on particular properties of sensorimotor actions related to the practice of counting.

This example, one we have studied quite thoroughly with many children, was first suggested to me by a mathematician friend who quoted it as the point of departure of his interest in mathematics. When he was a small child, he was counting pebbles one day; he lined them up in a row, counted them from left to right, and got ten. Then, just for fun, he counted them

<sup>5</sup> A more recent literature on embodied cognition [Lakoff & Núñez, 2000; Radford, 2014] argues for the general importance of embodied action in cognitive and developmental processes.

Name	Definition	Example
Subunit (Lesson 10)	Dividing a unit into <u>equal</u> distances creates subunits.	
Length of the subunit (Lesson 10)	The more subunits in a unit the shorter the subunits are.	
Denominator (Lesson 11)	The number of subunits in a unit.	
Numerator (Lesson 11)	The number of subunits.	
Fraction (Lesson 11)	$\frac{\text{numerator}}{\text{denominator}}$	
Mixed number (Lesson 13)	A whole number and a fraction.	
Whole numbers as fractions (Lesson 14)	A whole number can be written as a fraction.	
Equivalent fractions (Lesson 15)	Fractions that are in the same place but with different subunits.	
Benchmarks (Lesson 18)	0, $\frac{1}{2}$ , and 1 are benchmarks. You can tell ABOUT how big a fraction is by comparing the numerator and denominator.	

**Fig. 2.** Number line definitions: fractions.

from right to left to see what number he would get, and was astonished that he got ten again. He put the pebbles in a circle and counted them, and once again there were ten. He went around the circle in the other way and got ten again. And no matter how he put the pebbles down, when he counted them, the number came to ten. He discovered here what is known in mathematics as commutativity, that is, the sum is independent of the order. But how did he discover this? Is this commutativity a property of the pebbles? It is true that the pebbles, as it were, let him arrange them in various ways; he could not have done the same thing with drops of water. So in this sense there was a physical aspect to his knowledge. But the order was not in the pebbles; it was he, the subject, who put the pebbles in a line and then in a circle. Moreover, the sum was not in the pebbles themselves; it was he who united them. The knowledge that this future mathematician discovered that day was drawn, then, not from the physical properties of the pebbles, but from the actions that he carried out on the pebbles.

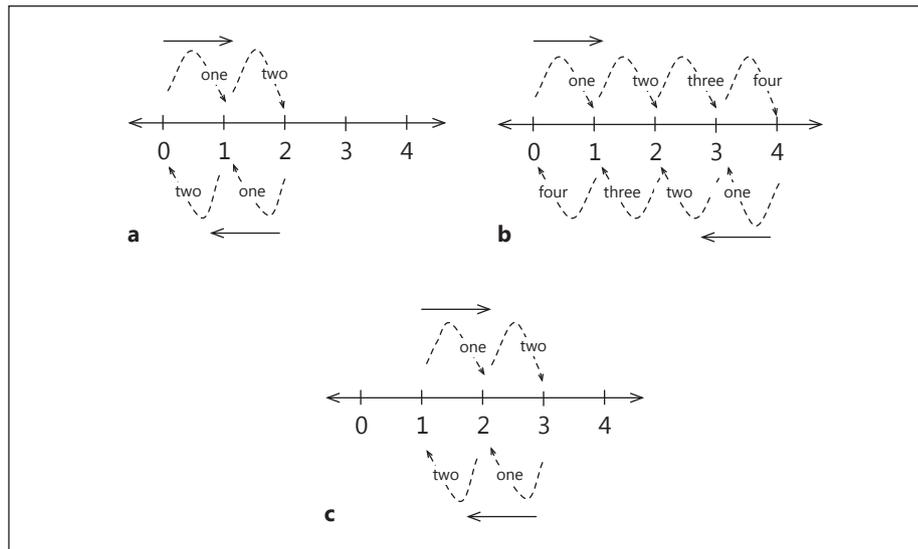
Of course, as Piaget indicated, the ability to use and reflect on schemes for sensorimotor actions is not unique to his mathematician friend or to counting actions applied to discontinuous quantities, like pebbles. Indeed, all children develop sensorimotor action schemes that may serve as resources for the construction of logico-mathematical knowledge across contexts – like the potential for counting pebbles with one ordering and then producing the same value by counting them in a different order [see Inhelder & Piaget, 1969]. Furthermore, when children extend such actions and logico-mathematical coordinations to a cultural artifact like the number line, actions can provide a resource for generating mathematical relations with the artifact and support mathematical problem solving.

We identify three action schemes that can serve as generative resources for students' construction of mathematical relations with the number line: counting, displacing, and splitting. These actions take on logico-mathematical properties as children construct varied kinds of reversibilities or mathematical closure, both properties of a mathematical group [see Beth & Piaget, 1974; Inhelder & Piaget, 1958]. The actions enable the treatment of the number line as a mathematical object, one in which intervals can be composed and decomposed. For example, children gain an understanding that as one counts tickmarks in one direction, one can return to the starting point by counting backwards the same number of marks. Further, as one displaces a line segment one unit to the right, one can reverse the direction one unit distance to the left and again return to the original position. Finally, as one splits an interval into parts, one can recompose the segments to achieve the whole of the original length. Children's reflections on properties of these actions and their coordination will figure centrally in our treatment of children's developing knowledge related to the number line.

### *Counting Actions*<sup>6</sup>

Like Piaget's mathematician friend, many children have engaged in counting activities involving discontinuous quantities prior to school entry [Fuson, 1988; Gelman & Gallistel, 1978; Sarnecka & Carey, 2008; Saxe, Guberman, & Gearhart, 1987].

<sup>6</sup> Though we identify "counting" as an action scheme in our analysis, we appreciate that actions are situated in sociohistorical contexts, and different communities have developed different representation forms for number [Chrisomalis, 2010; Lean, 1992; Menninger, 1969; Saxe & Posner, 1982]. We might more properly refer to counting actions as the production of ordinal and/or cardinal one-to-one correspondences between cultural forms of representation and to-be-represented objects [see Gelman & Gallistel, 1978; Saxe, 1977, 2012].



**Fig. 3.** **a** Counting two consecutive integers beginning with 0 and then reversing the count and ending at starting point. **b** Counting four consecutive integers from 0 and then reversing the count, ending at starting point. **c** Counting two consecutive integers from 1 and then reversing the count, ending at starting point.

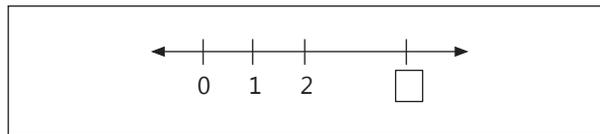
Such counting actions may be used as a resource to explore logico-mathematical properties attributed to the number line [Bofferding, 2014; Fosnot & Dolk, 2001]. Imagine, for example, a child counting tickmarks beginning at 0 and counting with two jumps to 2, and then negating the addition with two counting pulses to 0 (fig. 3a), perhaps reflecting that the second count of two undoes the first. A child might extend the exploration with a count to four (fig. 3b) or begin with different starting points (fig. 3c). Through the reflection of such counting actions, students may produce a generalization: the addition of any value can be inverted by subtracting the same value.

Children’s application of counting actions could be used generatively to conceptualize and solve number line problems. For example, imagine a problem like that depicted in figure 4 – a line with some tickmarks labeled and a student required to label the unlabeled mark. Clearly, counting could be drawn upon as a resource to conceptualize such a number line problem and to generate a solution. However, as we point out shortly, the solution may capture only a coordination of order relations, not the construction and coordination of metric linear units.

### *Displacing Actions*

In addition to counting, young children engage in activities outside of the classroom in which they displace objects from one position to another, like sliding a stick. In such activities, a child may reflect on logico-mathematical properties of actions,

**Fig. 4.** Labeling a marked point on the number line partitioned into intervals of unequal lengths.



**Fig. 5.** Number line problem: Find the position of 7 on the number line.



like the fact that moving an object from one location to another can be negated by reversing the movement to its starting point, or invariant properties of physical materials, like the conservation of length – the length of the stick is invariant as it is re-located in space [Piaget et al., 1960].

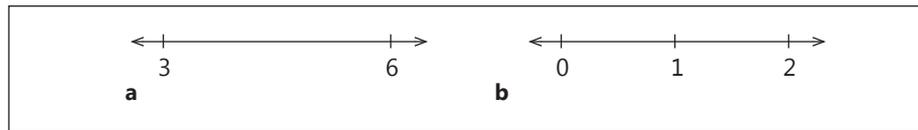
How might the reflective coordination of displacement actions be extended to conceptualizing and solving problems on the number line? Imagine, for example, a student presented with a number line problem contained in figure 5: Find the position of 7 given the positions of 2 and 3. One approach to solving the problem could be through a coordination of displacing and counting actions. For example, a block the length of the unit interval could be iteratively displaced and coordinated with counting actions to identify the position of 7. These coordinated actions may also lead the child to the inverse entailment corroborated by further exploration: that the action can be inverted to lead to the starting point – five displacements to the left of 7 returns to the starting point.

As with all sensorimotor actions, displacement actions take on logico-mathematical properties insofar as they are conceptualized as elements in a closed system; the additive composition of one length to another leads to a third in the system, and the act of composition can be decomposed through the inverse operation. To engage with such actions would require that children conceptualize the length of the block as invariant as it is translated from one position to another, end-to-end [Piaget et al., 1960].

### *Splitting Actions*

Subdivision [Piaget et al., 1960] or splitting [Confrey & Smith, 1995; Steffe & Olive, 2010] is the final class of actions we review that has logico-mathematical properties relevant to the number line. In their daily activities, children may produce fair shares and show rudimentary knowledge of splitting, whether in sharing discontinuous quantities [Davis & Pitkethly, 1990] or continuous quantities [Empson, 1999; Hunting & Sharpley, 1988; Pothier & Sawada, 1983].<sup>7</sup> As with displacing and count-

<sup>7</sup> Authors differ somewhat in their technical use of the term “splitting,” and Steffe and Olive [2010] have elaborated at some length how his use of the term differs from that of Confrey’s. We make clear our meaning of the term in the paragraphs to follow.



**Fig. 6. a** A number line with the interval from 3 to 6 labeled with the task to identify the position of 4. **b** A number line with labels for 0, 1, 2 with the task to identify the position of  $\frac{3}{4}$ .

ing actions, children may observe that splitting actions can be reversed. For example, a collection of candies can be split into groups and then reassembled, or a stick may be split and then returned to its original configuration.

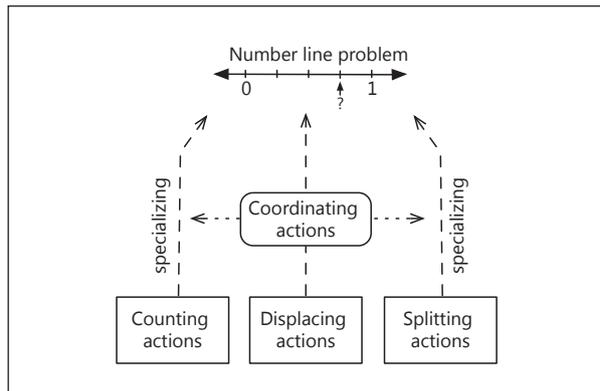
Splitting actions can be central to the identification of both integers and fractions on the number line. Consider, for example, the problem depicted in figure 6a: Find the position of 4 given the location of 3 and 6. To support a solution with the knowledge that 4 is between 3 and 6, students could draw upon splitting to divide the line several times in order to estimate the appropriate point. Or, consider the problem involving rational numbers in figure 6b: Locate the position of  $\frac{3}{4}$  on the number line that is labeled with 0, 1 and 2. In such a case, subdividing the line through successive splits would be an approach that would give access to the construction of numbers between other numbers.

### **Actions, Definitions, and Solving Number Line Problems**

We now provide a framework that is intended to capture relations between actions and definitions as students conceptualize and solve number line problems. The framework has two principal foci. The first consists of children's extension of actions to conceptualizing and solving number line problems and the challenges that children face in this process. The second focuses on children's construction of action-definition relations: children's use of actions in making sense of definitions as well as children's use of formal definitions to regulate actions. In a later section, we will apply the framework to a longitudinal analysis of a classroom community.

#### *Drawing on Sensorimotor Actions to Solve Number Line Problems: The Challenge of Specialization and Coordination*

Each kind of action and its negation – whether in counting, displacing, or splitting – is pertinent to constructing solutions to mathematical problems on number lines. We discuss two classes of adaptations that children must make in order to successfully perform this extension. The first is related to the specialization of schemes for sensorimotor action to accommodate the conventions and material properties of the number line. The second challenge is related to the coordination of actions; the focus is on the way different kinds of actions are used in concert to solve number line problems. We have highlighted processes of specialization and coordination of actions in problem solving activity in figure 7, and we note issues related to each below.

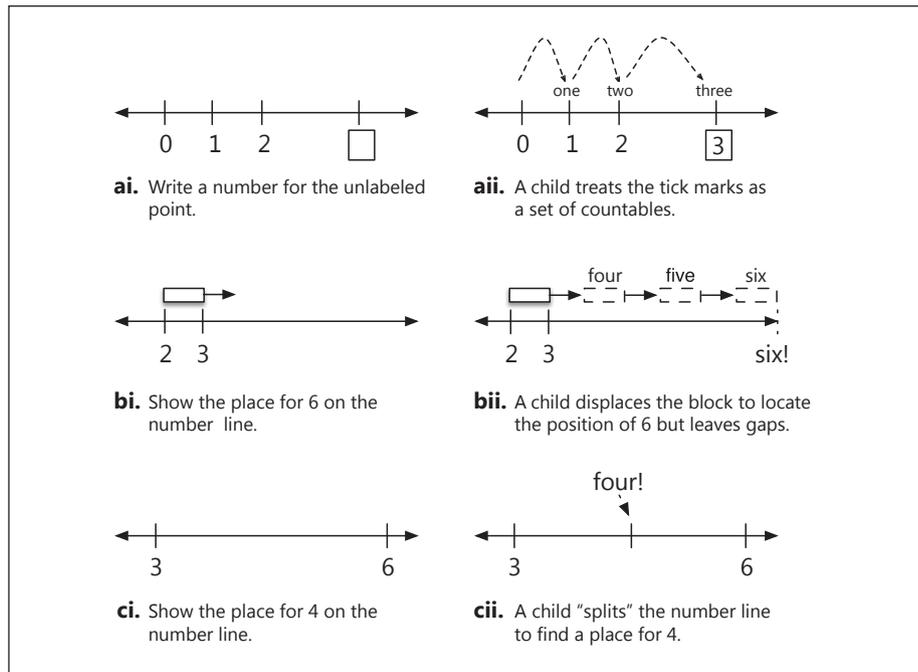


**Fig. 7.** Actions specialized and coordinated as they are used to solve a number line problem.

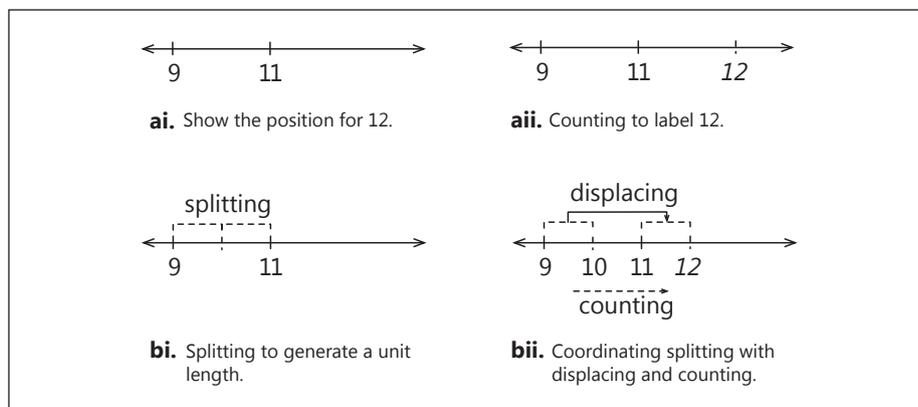
Consider problems of specialization. In order for actions to be useful, children need to specialize them for the number line. This specialization has two core facets. First, children must perform actions in a way that respects the ordering of numbers on the line. By convention, the number line is oriented horizontally, and numbers increase in value from left to right. Thus, counting towards the right or displacing towards the right implies an increase in numerical value, and vice versa. Similarly, the positions generated through splits take on value with respect to the left-right axis of the line.

Second, children must specialize these actions in ways that treat the number line as representing a continuous linear dimension that can be partitioned into metric units. This kind of specialization often presents difficulties for children. Indeed, children's actions often reflect their treatment of the line as comprised of discontinuous elements (e.g., tickmarks) rather than continuous lengths [Piaget, 1952; Saxe et al., 2013b]. Thus, in counting actions, a child may count tickmarks in conceptualizing and solving a problem like the one depicted in figure 8ai, identifying the missing value incorrectly as “3,” not “4” (fig. 8aii). In displacing actions, a child may use a rigid block as a measure for length (fig. 8bi), but displacing actions may be carried out in ways that do not respect length as a continuous dimension, leaving gaps (fig. 8bii). In splitting actions, a child can generate positions for missing values by creating subdivisions, as when a child is asked to identify the position of 4 on a line with 3 and 6 labeled (fig. 8ci); however, the child's splits may not well coordinate linear distance with numerical values, splitting at the midpoint between 3 and 6 to show 4 (fig. 8cii) or creating two splits, one to show that 4 is near 3 and another to show that 5 is near 6 [Piaget, 1952; for an extended argument and empirical support, see Saxe et al., 2013b].

Next, consider problems of coordinating actions in the course of a problem solution like that depicted in figure 9ai [Saxe et al., 2013b]. In the problem, a child is asked to place 12 on a number line with 9 and 11 labeled. Through counting, a child may position the number 12 to the right of 11 with an interval that is the same as that between 9 and 11 (fig. 9aii). What the child has not accomplished in this labeling is successfully coordinating actions – for example, splitting the interval from 9 to 11 into unit intervals and then using one of those unit intervals to displace and count one

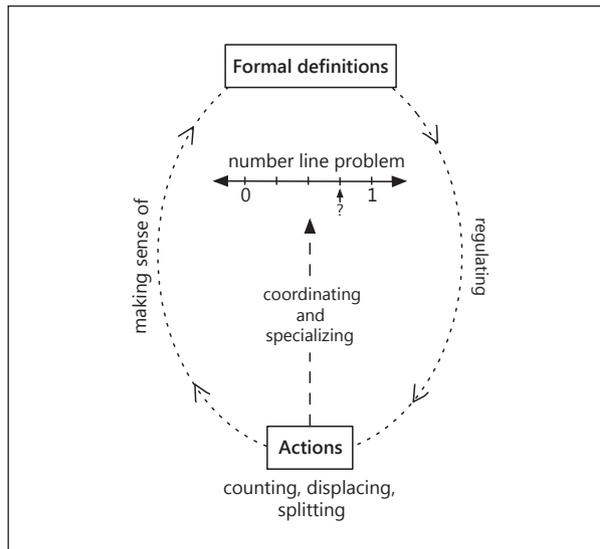


**Fig. 8.** The specialization of counting (a), displacing (b), and splitting actions (c) in which length is treated as a discontinuous quantity.



**Fig. 9.** The coordination of counting, displacing, and splitting actions in labeling a position for 12 on a number line. **a** The problem and a counting solution. **b** A solution that coordinates splitting, counting, and displacing actions.

**Fig. 10.** The potential for a productive interplay between definitions and actions: actions used as a resource for making sense of formal definitions; formal definitions used as a resource for regulating actions as they are coordinated and specialized to solve a number line problem.



linear unit to the right of 11, in order to label 12 (fig. 9bii); instead, the child has placed 12 in the position of 13 (fig. 9bi).

We have pointed to the way in which students' counting, displacing, and splitting actions can provide footholds into solving number line problems, opening up new developmental challenges. As they begin solving number line problems, students may specialize actions to accommodate the ordering of numbers on the line, but they may not initially treat the line as a continuous linear dimension. New challenges emerge when students begin treating the number line as a representation of continuous quantities, requiring metric units of length and a coordination of multiple actions.

*The Potential for a Productive Interplay between Actions and the Register of Definitions: Bottom-Up and Top-Down Relations*

So far, our focus has been students' specialization and coordination of actions in the process of conceptualizing and solving number line problems, challenges that are schematized in figure 7 and reproduced in the inner oval of figure 10. We now turn to LMR's mathematical register that indexes formal definitions. We take up the question of how the use of actions might support children's efforts to make sense of definitions (bottom-up processes, the left part of the oval's perimeter in fig. 10). We also consider the question of how children's developing sense of definitions might, in turn, support the regulation of actions (top-down processes, the right part of the oval's perimeter in fig. 10). We elaborate these ideas below.

*Bottom-Up: Actions as Resources for Making Sense of Definitions.* How might a student come to make sense of any of the varied definitions introduced for integers (fig. 1) and fractions (fig. 2) in whole-class discussions? While memorizing defini-

tions and associated terms may be helpful for some purposes, memorization alone leads to what we noted that Whitehead [1959] referred to as “inert knowledge.” To make mathematical meaning of a definition requires going beyond its surface, transforming it into systems of meanings or actions.

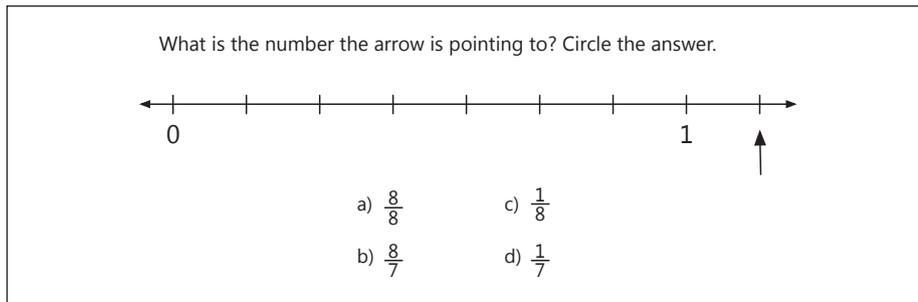
A student can transform surface properties of definitions through a wide range of action coordinations. These action coordinations include translating a formal definition into a representation on the number line. For example, a student might understand the formal definition of a *unit interval* (“the distance from 0 to 1 or any distance of 1”) as a 0-to-1 interval. In order to make sense of “any distance of 1,” a student might use a coordination of displacing and counting actions to identify additional distances of 1 on the number line. Furthermore, a child might explore generative uses of the *unit interval* definition, displacing the 0-to-1 segment to unmarked regions of the number line, and consider numerical labels for newly identified points. Finally, the child might also consider incongruities – for example, that two different number lines may have different scales, so that the length between two consecutive numbers on one scale would differ for another scale.

Students’ developing understandings of newly introduced definitions can also have implications for their understandings of previously introduced definitions. For example, the introduction of *multiunit interval* (the multiple of a unit interval) may lead to a new regard for the affordances of the definition of a *unit interval*. With respect to multiunits, units are their constituent parts, and units, in turn, may be composed into new multiunits with similar properties (e.g., they can be displaced to different positions on the line). Likewise, the definition of a subunit in the later fractions lessons supports new functions for a unit interval. With the definition of a subunit, the unit interval takes on the new function of providing boundaries for equal partitions of a unit interval.

*Top-Down: The Register as a Resource for Regulating Actions.* In a classroom implementing the LMR curriculum, students are participating in a discourse community that privileges a register of mathematical definitions. These definitions offer students resources for regulating actions. To illustrate, consider the potential role of the definition of *denominator* while solving the fractions problem depicted in figure 11. In this problem, students are required to represent a fractional value that extends beyond the integer 1; the only correct choice from several alternatives is an improper fraction. In solving such a task, many students will treat each smaller interval as a subunit, counting 8 as the number of subunits, and, in turn, represent the targeted point with a numerator of eight but also a denominator of eight (i.e.,  $\frac{8}{8}$ ). Recruiting the definition for *denominator* (the number of subunits in the unit) can support students in regulating how they count parts of unit intervals, leading to the identification of the target point as  $\frac{8}{7}$ , not  $\frac{8}{8}$ .

### **A Longitudinal Analysis of the Classroom Community: The Interplay between Actions and Definitions in Collective Practices**

We now undertake a longitudinal analysis of a single classroom community as it enacts LMR’s 19-lesson sequence, with a focus on the interplay between actions and definitions. As context, we note that the fifth-grade classroom selected for analysis



**Fig. 11.** Improper fractions task: using a definition to regulate the construction of subunits.

was located in a public elementary school in the San Francisco Bay area, and it contained students from a range of socioeconomic levels and of varied ethnicities.<sup>8</sup> The teacher, whom we refer to as Mr. K, had more than 20 years of upper elementary teaching experience; he had some familiarity with LMR lessons prior to the year of data collection, having piloted selected lessons in the year prior to data collection. Noteworthy also is that the video and other forms of data collected in Mr. K’s classroom were part of an efficacy study of the LMR curriculum; like other LMR classrooms in the study, the students in Mr. K’s classroom made strong gains relative to control classrooms on measures of integers and fractions knowledge [Saxe et al., 2013a].

To bound and organize our analysis, we selected “collective practices” as a principal arena. Collective practices are reproduced and altered in interactional activity as norms, conventions, social positions, and even mathematics become constituted and re-constituted in interaction [Saxe, 2012]. We guided our analysis by two questions related to the interplay between actions and definitions in illustrative collective practices. The first was how might the teacher have introduced definitions over the course of lessons in ways that support sense making of the definitions through actions? The second was how might the teacher have supported sustained references to definitions such that definitions would be more likely to enter into students’ conceptualization of both number line problems and the actions they used to solve them? Though we frame these questions with an emphasis on the teacher, as we shall see, the students in the classroom became participants in both the introduction and sustained use of definitions in collective practices.

#### *Timeline: A Bird’s-Eye View of Definition Use*

Our video corpus included recordings of all 19 lessons. The many lessons translated into 23 hours of video recording, making data reduction a challenge. Our approach began with a selective focus on classroom talk, coding references to defini-

<sup>8</sup> Twenty-eight students in total (15 boys, 13 girls): 39% Caucasian, 36% African American, 11% Latino, 7% Asian, and 7% multiethnic.

tions in whole-class discussions. The coding provided a basis to construct a “definitions timeline,” a bird’s-eye view of definition reference over the entire lesson sequence. Using the timeline as a rough guide, we were particularly interested in determining the extent of references to definitions over the lesson sequence, identifying first occurrences of references that might signal their introductions, and identifying collective practices that supported the sustained use of definitions over lessons.<sup>9</sup>

We represent the results of the coding in the timeline in figure 12. The timeline represents the 19 lessons sequentially from left to right, including nine integers lessons (the first set focusing on negative and the second on positive integers) and ten fractions lessons (the first set focusing on part-whole relations and the second on multiplicative relations). The leftmost column contains definition names that comprise the definition register.<sup>10</sup> Each hatch mark represents a single reference to the respective definition through the lessons in the classroom community. We included 16 of the 17 definitions in our coding. References to the “fractions” definition were excluded, due to redundancy with definitions for “numerator” and “denominator.”

Several features of the timeline informed our analytic approach. First, definition expressions were used frequently over the lessons in whole-class discussions, so a qualitative analysis seemed promising. Indeed, we counted more than 1,400 instances. Second, as expected, the timeline allowed us to locate the introduction of each definition, which became a basis to identify collective practices in which definitions were introduced. Third, the timeline provided a “map” of recurring references to definitions, which supported our identification of collective practices that contributed to recurring use.

### *Collective Practices and the Definitions Register*

We identified two collective practices in whole-class discussions that became the crux of our analytic focus. As will become clear, these practices present microcosms constituting small but important units of classroom life. They are each marked by the reproduction of norms, social positions, and patterns of interaction as the use of definitions and actions become realized in activities. Consistent with our analytic goals, we identified one collective practice that frequently occurred when terms in the definitions register were introduced, a collective practice that we refer to as *Defining*. The second collective practice involved a pattern of joint activity that sustained use of the register after terms had been introduced, a collective practice that we refer to as *Correcting the Teacher*. We note that, while these collective practices are consistent with design principles that guided the construction of the LMR curriculum, Mr. K’s use of *Defining* and *Correcting the Teacher* practices were a part of Mr. K’s personal instructional approach.

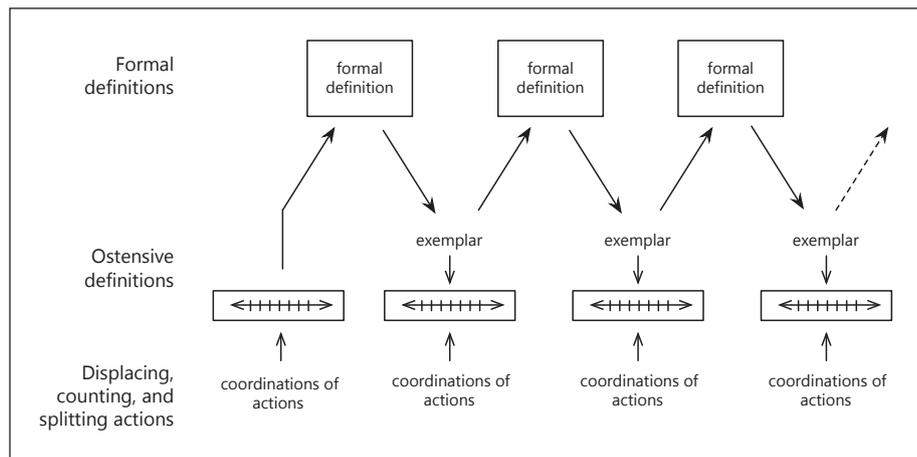
*The Collective Practice of Defining Terms in the Mathematical Register: The Joint Production of Ostensive and Formal Definitions.* Because the LMR curriculum provided a register of 17 definitions, we were able to consider whether there were iden-

<sup>9</sup> For our data reduction that supported the generation of a timeline, we made use of the video coding software StudioCode.

<sup>10</sup> The top-to-bottom order of the definition names reflects the order in which they are introduced in the lesson guide.

Definitions and principles	Integers lessons										Fractions lessons								
	Positive integers					Negative integers					Part-whole relations				Multiplicative relations				
	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12	L13	L14	L15	L16	L17	L18	L19
Order																			
Zero is a number																			
Interval																			
Unit interval																			
Multunit																			
Every number has a place																			
Symmetry																			
Absolute value																			
Subunit																			
Length of subunit																			
Numerator																			
Denominator																			
Mixed number																			
Whole number as fraction																			
Equivalent fractions																			
Benchmark																			

Fig. 12. Timeline showing references to definitions by students and teacher over the 19 lessons.

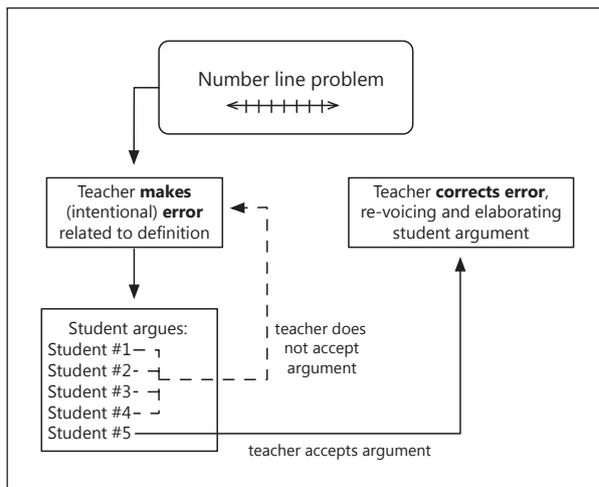


**Fig. 13.** The collective practice of *Defining*: A teacher orchestrating the class in a back and forth between formal definitions, ostensive definitions, and logico-mathematical actions on the number line.

tifiable collective practices in which definitions were introduced. Figure 13 contains a schematic of one such practice that we refer to as *Defining*, a pattern of interaction that was often used to introduce terms of the register. A core feature of the *Defining* practice is the back and forth between teacher and students as participants move between what we refer to as “formal definitions” and “ostensive definitions.” Formal definitions are statements that *stipulate* the scope to which a definition term is meant to refer, establishing criteria for what should be included and excluded (see middle column of (fig. 1, 2). For example, the formal definition of *unit interval* stipulates that “it is the distance between 0 and 1 on a number line or any other distance of 1.” Ostensive definitions are *demonstrations* of meanings, often generated by pointing or other ways of showing the meaning of a term with direct reference to its object [Wittgenstein, 1953], like gesturing to a bracketed 0-1 distance on a number line and saying “that’s a unit interval.” In the context of instruction, we found that the number line inscriptions were often produced in teacher-student interactions with displacement, counting, or splitting actions, or a coordination of two or more of these actions.

*The Collective Practice of Correcting the Teacher: Drawing on Definitions to Support an Argument.* We noted earlier that the timeline (fig. 12) revealed that often, after a definition was introduced, that same definition was referred to many times over in subsequent lessons, sustaining a semiotic context of definition use. This prompted us to ask whether there were collective practices in Mr. K’s class that led to the sustained use of the mathematical terms over lessons. Further, did these collective practices provide support for the regulation of actions with definitions? We identified a second focal collective practice, *Correcting the Teacher*, that served these functions. As we show, this collective practice accounts for some of the extended uses of definitional terms and provides occasions for supporting definition use in conceptualizing and solving number line problems.

**Fig. 14.** A schematic of the *Correcting the Teacher* collective practice.



Like the practice of *Defining*, the collective practice of *Correcting the Teacher* is constituted by a pattern of teacher and student talk. To initiate the *Correcting the Teacher* practice, Mr. K feigns an error in solving a number line problem, an error that violates a mathematical definition or its entailment (fig. 14). This teacher move positions the class to evaluate the teacher’s claim. Students respond by agreeing with or challenging the teacher’s claim, often by appealing to mathematical definitions. If students do not justify their claim with reference to definitions, the teacher continues to re-assert his (erroneous) claim until he hears a student reference a mathematical definition or mathematical idea in a potentially productive way. Upon hearing a satisfactory response, the teacher re-voices the student’s justification, elaborating the relevant definition and its relevance to the problem under discussion.

#### *Six Episodes: From Integers to Fractions*

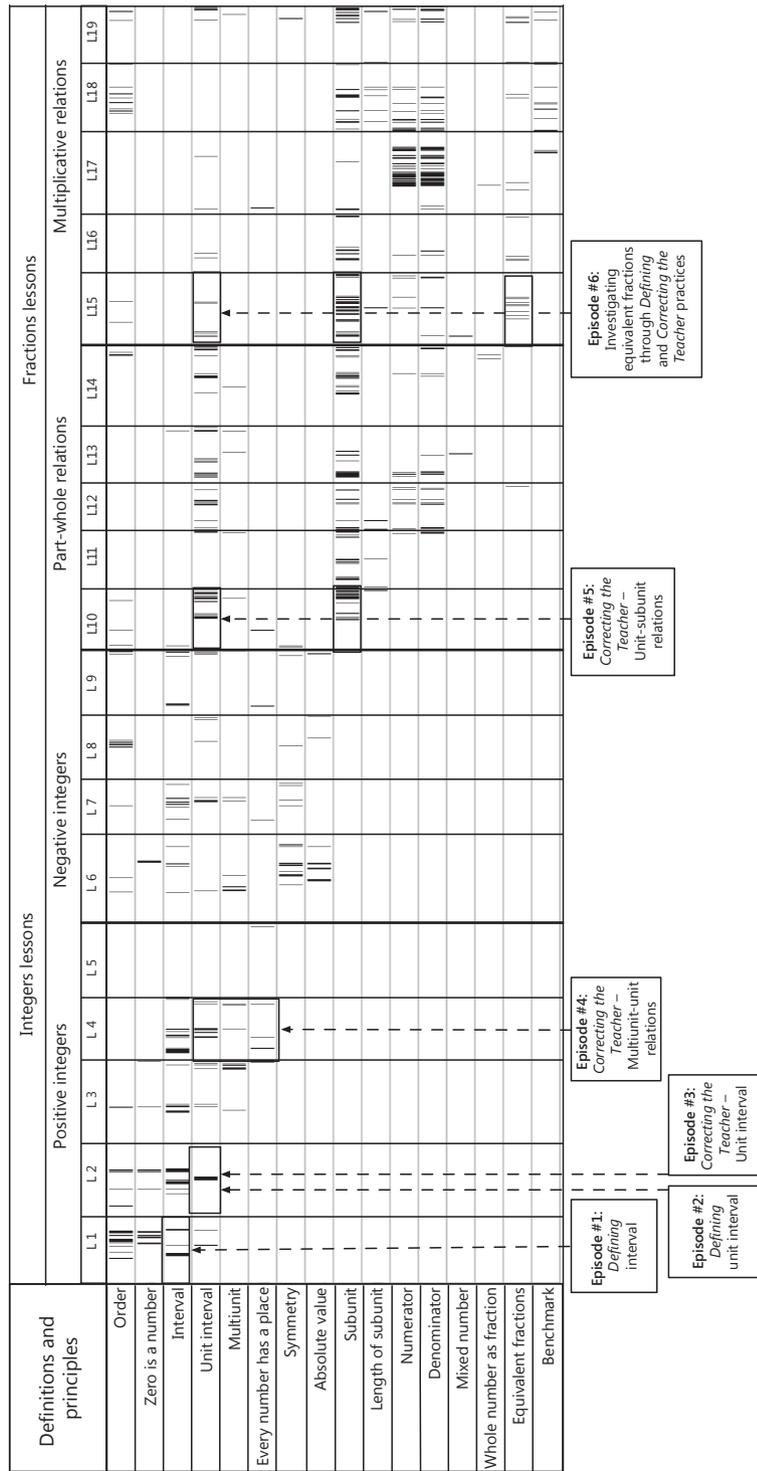
To illustrate the interplay between actions and definitions over the lesson sequence, we draw on six collective practice episodes that occurred in Mr. K’s classroom. In our selection of episodes, we sampled from both integers lessons (lessons 1 through 9) and fractions lessons (lessons 10 through 19). Among the many episodes we found of *Defining* and *Correcting the Teacher*, we further constrained our selection by considering only those that featured references to at least one of five definitions: *interval*, *unit interval*, *every number has a place*, *subunit*, and *equivalent fractions*. Our choice of these particular definitions was guided by several ideas. First, definitions were constitutive of the number line as a continuous mathematical object, partitionable into metric segments (*interval*, *unit interval*, *multiunit interval*, *subunit*, *equivalent fractions*) or a principle central to understanding properties of the line (*every number has a place*). Second, the definitions we selected showed logical interdependencies with one another, so that one could derive some definitions from the entailments of others (e.g., *unit interval*, *multiunit interval*, and *subunit* are all kinds of

*intervals*). A third feature that we wanted represented in our analysis was “long life,” as we were particularly interested in definitions that were used across multiple lessons in the timeline.

Figure 15 situates the six episodes in our timeline; each episode is labeled with the collective practice targeted for analysis. The associated arrows point to particular definition references that were foci within these analyses. Over the course of the episodes, our goal was to consider longitudinally the elaboration of the definition use in talk with special regard for emergent relations between definitions and actions. In collective practices of *Defining* (episodes #1, #2, and #6) a focal concern was how actions and previously introduced definitions provide students with resources to make sense of new definitions. In collective practices of *Correcting the Teacher* (episodes #3, #4, #5, and #6) a focal concern was to understand the supports for and use of definitions to regulate actions.

*Episode 1: Defining Interval.* Our first episode is drawn from lesson 1 of the integers series, a lesson conducted shortly after the start of the school year. In this episode, Mr. K leads a whole-class discussion of an early instance of the collective practice of *Defining* in his introduction of a definition for *interval*. As noted previously, the *Defining* practice is marked by an interplay between the production of ostensive and formal definitions. Teacher and student actions produce inscriptions (e.g., intervals on a number line) for what becomes pointed to as ostensive definitions, a process that provides support for the articulation of formal definitions (a definition of *interval*). Mr. K sits in a chair in front of the whiteboard and students are gathered close by on the rug. To initiate the *Defining* practice, Mr. K draws a number line from 0 to 6 on the class’s whiteboard (fig. 16), and frames the ensuing episode with the statement, “Let’s talk about the distance between numbers.” Mr. K brackets the distance between 1 and 4 on the number line (with pinched fingers), asking, “What’s the distance between 1 and 4?” (as depicted in fig. 17a). He then clarifies the question in a way that affords the coordinated use of displacing and counting actions: “If you were walking from 1 to 4 (pointing iteratively to each tickmark between 1 and 4) and a step got you 1 of these tickmarks, how far would you walk?” (fig. 17b). To confirm students’ answers of 3, Mr. K counts aloud as he gestures three hops with the tip of his finger from 1 to 4 on the number line (fig. 17c). He concludes with an ostensive definition linked to the prior displacing and counting actions, “That is an *interval* [emphasis] of 3,” and labels the bracket with a 3 (fig. 17d). He then reasserts the ostensive definition, emphasizing the importance of the register, “1 to 4 is an *interval* of 3; that’s another big word that you need to know.”

Having established an object to serve as an ostensive definition – a bracketed interval of 3 (fig. 17d) – Mr. K segues to a formal statement, asserting, “An interval is a distance on the number line. Any distance is an interval.” Then, typical of his back and forth movement in the *Defining* practice, he shifts back to an ostensive definition: “This interval happens to be an interval of 3.” Next, Mr. K extends the *Defining* practice by posing new questions about values of intervals, bracketing new intervals on the line, and asking the class to call out their values, a process that affords the use of actions to generate interval values. In the course of this interaction, the class constructs new ostensive definitions for *interval*, including intervals of 2 to 3, etc. Mr. K concludes with a restatement of the formal definition that builds on ostensive definitions and the actions that enabled them: “So an interval is just any distance on the

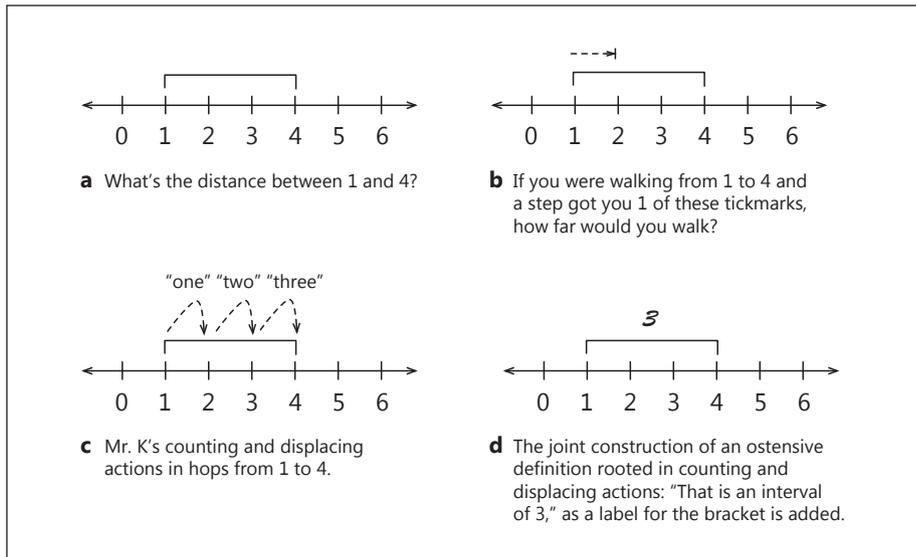


**Fig. 15.** Reproduction of the timeline with six targeted collective practice episodes highlighted. Labels indicate episode number, collective practice type, and arrows indicate definitions referenced.



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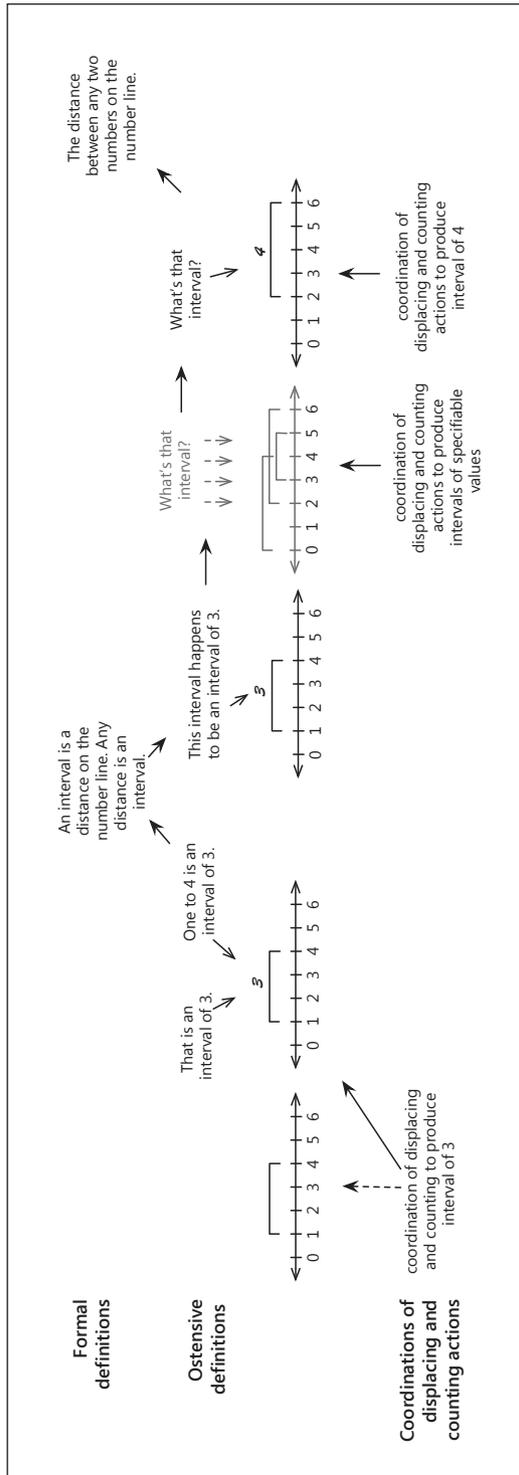
**Fig. 16.** Mr. K and his class, with the 0 to 6 labeled line on the whiteboard.



**Fig. 17.** The construction of an ostensive definition through coordination of counting and displacing actions with specialization to linear distance.

number line ... An interval is *any* distance between two numbers, we call that an interval."

Figure 18 contains a schematic portrayal of the *Defining* episode for *interval*. In the back and forth between the production of ostensive and formal definitions for *interval*, Mr. K affords his students multiple opportunities to use counting and displacing actions to make sense of his own (and one another's) displays of both the



**Fig. 18.** A back and forth between logico-mathematical actions, ostensive definitions, and formal definitions in the *Defining* practice for interval.

formal definition and the ostensive definitions. In his displays such as, “If you were walking from 1 to 4 and a step got you 1 of these tickmarks, how far would you walk?” (counting aloud as he gestures three hops from 1 to 4 on the number line), Mr. K roots the numerical value of an interval in coordinated displacing and counting actions. The reference to displacing actions is contained in Mr. K’s reference to the student walking, and the reference to counting actions is contained in the query about how many steps, with the accumulation (count) of steps producing a cardinal number. Many students seem to be taking advantage of these opportunities for counting and displacing actions, reporting correct values to Mr. K when he asks them to specify interval values on the line. Their engagement provides resources for reflection on the meaning of Mr. K’s formal definition: “An interval is a distance on the number line. Any distance is an interval.” They would be likewise prepared to reflect on the formal definition that would later be inscribed on the classroom poster: “The distance between any two numbers on the number line.”

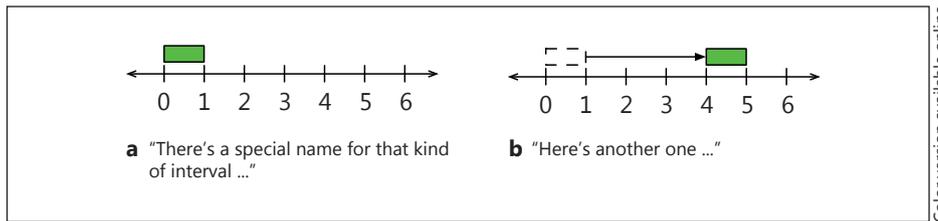
While the teacher’s use of actions on the number line may support students’ reflections, it is not clear how children are actually making sense of the definition in relation to these actions. We take the displays that Mr. K was producing and orchestrating as amenable to multiple interpretations, and we note that diverse conceptualizations may be consistent with correct answers to Mr. K’s queries. For example, some students may be conceptualizing the line as consisting of discontinuous quantities of countable tickmarks, where *interval* refers to a collection of tickmarks. Others may be treating the definition in a manner consistent with Mr. K’s intentions: as referencing a metric space, one in which length is central to the definition.

*Episode 2: Defining Unit Interval.* In lesson 2 (integers), we find another instance of the *Defining* practice in which the focus was on a new register term, *unit interval*. In this instance, Mr. K grounds the introduction of *unit interval* in the prior definition of *interval*. Mr. K asserts, “there’s a special kind of interval ... it’s the most important that we’ve gotta learn on the number line; it’ll help us unlock the secrets of all number lines.” He then draws attention to the 0-to-1 distance by placing a Cuisenaire™ rod<sup>11</sup> that fits between 0 and 1 on top of the line (fig. 19a), asking, “How much is that interval?”, whereupon the class responds in chorus, “one.” In rapid sequence, he then creates an ostensive definition that leads to his articulation of a formal definition. Mr. K says, “There’s a special name for that kind of interval ...” He writes “Unit Interval” on the whiteboard and states, “It means a distance of one.” Mr. K then generates a new ostensive definition. Referring to the 0-to-1 distance, he says, “So that’s a unit interval,” and then *displaces* the Cuisenaire rod to the 4-to-5 interval on the line, saying “Here’s another one” (fig. 19b).

Mr. K then challenges the class to explain why both are unit intervals, and students respond that they are both distances of 1. Mr. K reiterates and illustrates their statements, capturing the 4-to-5 interval with a finger pinch and displacing it back to the 0-to-1 position. He asks the students to “turn and tell a partner”<sup>12</sup> about ad-

<sup>11</sup> Cuisenaire rods are of ten lengths coded by color with each rod being a multiple of the shortest length rod (white). One red is equal in length to that of two whites; one light green is equal in length to that of 3 whites; one purple is equal in length to that of 4 whites, etc.

<sup>12</sup> Mr. K frequently elicits partner talk as a participation format. This has a number of affordances. For example, it allows a large number of students to voice their ideas and to obtain feedback from their peers, while allowing Mr. K to accomplish some selective monitoring.



Color version available online

**Fig. 19.** Ostensive definitions of unit interval with displacing action.

ditional instances of unit intervals on the line. After productive peer-to-peer chatter, Mr. K asks the class to think of additional unit intervals not on the whiteboard line – unit intervals “that you could imagine” – and asks students to share their answers with the whole class. A brief transcript is included below.

Teacher:	That's also a unit interval. Why?
Students (in chorus):	It's a distance of one.
Teacher:	(nods) It's a distance of one. It's the same distance as the distance between zero and one (displaces Cuisenaire rod back to the 0-to-1 interval on the line), they're identical (displaces rod back to the 4-to-5 interval on the line). That's also a unit interval (displaces rod to the 3-to-4 interval on the line). Turn and tell a partner another one you see on here.
Student (audible chatter):	Five and six.
Student (audible chatter):	Two and three.
Teacher:	More unit intervals please. Okay, now turn and tell somebody on the other side one (unit interval) that's <i>not</i> on this number line, that you could imagine.
Student (audible chatter):	Sixteen and seventeen.
Student (audible chatter):	One hundred and one hundred and one.
Student (audible chatter):	One million and one million and one.

Mr. K then walks to the poster of definitions and proceeds to call on each student individually (in the entire class) for an example of a unit interval that is not on the line.

The *Defining* practice ends with the introduction of the new term inscribed on the classroom poster (fig. 20a), accompanied by its formal definition: “Any distance from 0 to 1 or an equivalent distance” (fig. 20b). In the rightmost column, Mr. K inscribes an image that serves as a basis for ostensive definitions (fig. 20c). He emphasizes that the lengths of unit interval on the line are “consistent” – a term he adds to the poster (fig. 20d).

Over the course of this episode, we note that there is both continuity and discontinuity with the prior *Defining* practice. The prior formal definition of *interval* (and related ostensive definitions) provides a resource to interpret the new definition of *unit interval*; furthermore, the actions of displacing with the Cuisenaire rod, as well as the objectified unit interval of a finger pinch, afford students the opportunity to construct and explore entailments of the formal definition. Indeed, Mr. K's coordinated displacements of a rigid rod and counts of 1 may afford some students' insight into the invariant (i.e., “consistent”) length of unit intervals on a given number line.

Principle Name	Definition	Example
Order	Numbers increase in value from left to right. Numbers decrease in value from right to left.	
0 is a number	0 is a number on the number line so it has a place!	
Interval	The distance between any two numbers on the number line.	
Unit Interval	Any distance from 0 to 1 or an equivalent distance.	

**Fig. 20.** Definitions register showing new term added (unit interval), formal definition for unit interval, emphasis on consistent distance, and ostensive definitions that support the formal definition.

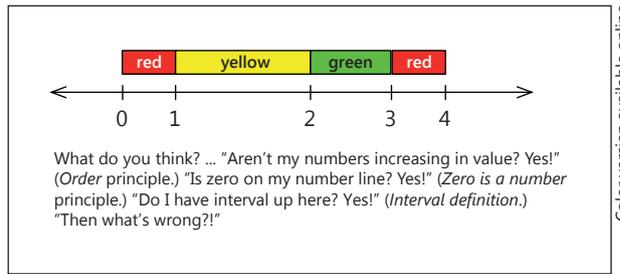
a Introduced definition register term  
 b Formal definition  
 c Ostensive definition  
 d "Consistent"

Yet, as Mr. K implies in his opening statement, we also find discontinuity in the introduction of the new term; *unit interval* is a new definition, one that has the potential to “unlock the secrets of all number lines.” We also note that some students may still not conceptualize unit intervals in a continuous metric space, an issue directly addressed in the next episode in which the class becomes engaged with a *Correcting the Teacher* practice.

*Episode 3: Correcting the Teacher – Unit Intervals.* In lesson 2 of integers, an occurrence of *Correcting the Teacher* brings to the fore the importance of conceptualizing distances on the number line as metric units of length (rather than as discontinuous tickmarks). This metric treatment of the number line calls for a further specialization of actions in which intervals are treated as linear metric units constituting a continuous linear dimension. The practice emerges after the completion of an activity in which students are instructed to build race courses that are four miles long with the red rod representing a distance of one mile. Mr. K organizes a reflective discussion about the activity through a *Correcting the Teacher* practice: He builds the 4-mile racecourse on the whiteboard with four Cuisenaire rods of different lengths and asserts that his racecourse is correct (fig. 21). As typical of the *Correcting the Teacher* practice, the class objects, and the discussion coalesces around the need to specialize and coordinate actions so as to respect an invariant linear unit.

The back and forth of teacher-student exchanges is extended in this instance of the *Correcting the Teacher* practice. We focus on key moments that illustrate the way the practice supports references to definitions as well as their use in regulating actions. After constructing the racecourse shown in figure 21, Mr. K asks the class, “What do you think?” and several students immediately reject Mr. K’s display. In a

**Fig. 21.** Mr. K’s 4-mile race-course, which he insists is “correct.”



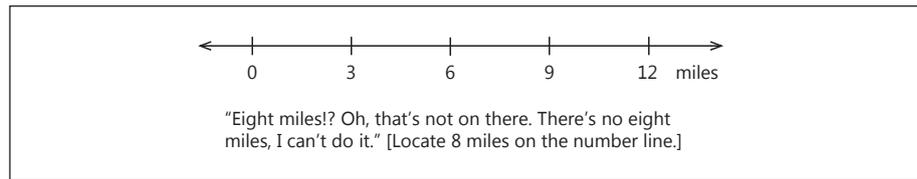
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display typical of *Correcting the Teacher*, Mr. K feigns surprise – “Why not!?” – and argues for the correctness of his race course. He continues voicing his argument over students’ repeated objections, appealing to several definitions that had been previously introduced, asking and answering several rhetorical questions: “Aren’t my numbers increasing in value? Yes!” (Order principle introduced in lesson 1.) “Is zero on my number line? Yes!” (Zero is a number principle, introduced in lesson 1.) “Do I have intervals up here? Yes!” (Interval definition, introduced in lesson 1.) “Then what’s wrong?!” He then instructs students to “turn and tell” a partner what is wrong with his racecourse.

After some partner talk, Mr. K states that he hears many students saying that the racecourse is wrong because the intervals are of different lengths, and he encourages them to articulate why this is a problem. Then, after some continued peer talk, Mr. K says he overheard the answer he was looking for. He calls on one student to share his reasoning, who says, “Because some of them are more than a mile.” Mr. K affirms the student’s answer and draws the class’s attention to the problem’s instructions:

I told you the red rod is 1 mile. You can’t all of a sudden make the yellow one mile. ... If I said on this number line the red is 1 mile, then the red is always one-mile. And here’s the word for that. It has to be (writes the word “consistent” on the board, which students read aloud in chorus). On any one number line, once you choose what it’s going to be, you got to keep it consistent. ... Now I could make a whole new number line where the yellow is 1 mile, right? And then all the yellows would be one-mile (makes a displacing motion of the yellow rod several times). ... *Consistency*, remember that word, it’s going to come up a lot.

In this instance of the *Correcting the Teacher* practice, we find that the definition of *unit interval* is referenced as a basis for regulating displacing and counting actions used in the construction of a number line with Cuisenaire rods. Though he does not specifically refer to the term “unit interval,” he references a term that he uses as a substitution for it: “consistent.” *Consistent* is a term that Mr. K inscribed on the classroom poster under *unit interval* and a term that he has come to use as a substitute for the reference to the definition (as have many students in the class). This episode also prefigures an idea that will be taken up in subsequent lessons – the idea that multiunit intervals can be constituted by the composition of multiple unit intervals, a composition rooted in successive displacing actions (and their decomposition afforded by splitting actions).



**Fig. 22.** The number line race that the class is considering on which 8 is not labeled.

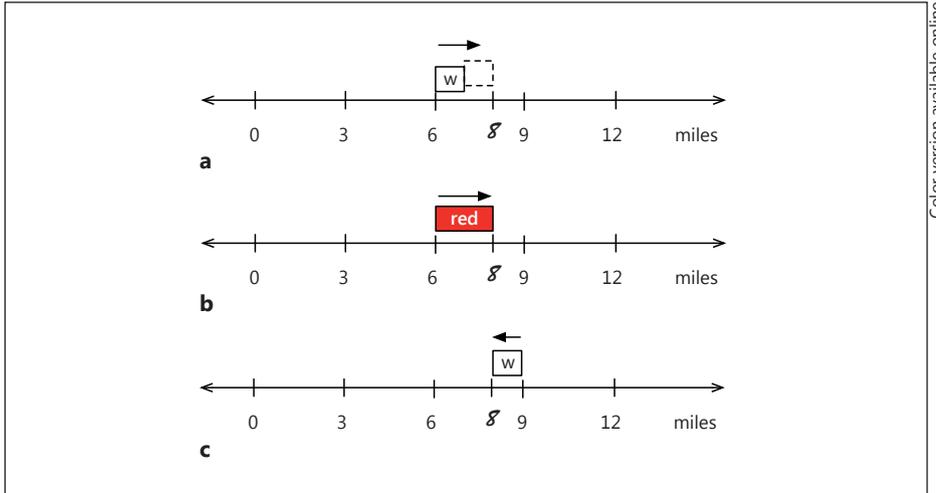
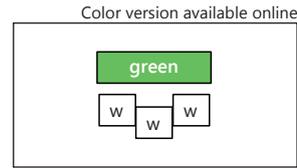
*Episode 4: Correcting the Teacher – Multiunit-Unit Relations.* During the fourth integers lesson, Mr. K again leads the class through a *Correcting the Teacher* practice that engages the idea of relations between unit intervals and multiunit intervals. The error he feigns prompts the students to draw on definitions introduced in prior lessons to formulate their arguments and to make connections across definitions.

A central mathematical idea that emerges in lesson 4 is that unit intervals can be composed to produce a multiunit interval, which can in turn be decomposed into unit intervals. Supporting the multiunit-unit relations idea are two terms of the number line register introduced in the prior lesson: *multiunit interval* (“a multiple of a unit interval”) and *every number has a place* (“every number has a place, but not every number needs to be shown”). To draw students into investigation of the new register term, *multiunit interval* and *every number has a place*, Mr. K engages the class with the problem of locating 8 miles on a race course with only 0, 3, 9, and 12 miles marked (fig. 22). Referencing the racecourse on the overhead screen, Mr. K asserts (incorrectly) to the class that 8 miles cannot be located on the line: “Eight miles!? Oh, that’s not on there.” A few students protest, but Mr. K continues, drawing a particular student’s attention to the apparent contradiction between the problem’s goal (to locate 8) and the appearance of the line: “There’s no eight miles, Luis; I can’t do it.” Not hearing anything from the students, Mr. K continues: “Zero, three, six, nine, and twelve; there’s no eight.”

In response to Mr. K’s mock insistence that 8 cannot be located, one student calls out loudly with “every number has a place!” and another calls out “you could use whites,” referring to the small white Cuisenaire rods. Taking up the first student’s call out, Mr. K repeats and ratifies the student’s mention of a relevant definition, noting the complete principle “every number has a place, but not every number needs to be shown.” Then Mr. K picks up on the idea of using white Cuisenaire rods to locate 8. After the class works through the observation that three whites fit within the green rod and the interval from 6 to 9 miles is 3 miles, they agree that one white is equivalent to one mile and, reciprocally, that a green can be split into three whites (fig. 23). Summarizing the student’s idea, Mr. K says, “Here’s what he did. He found the multiunit and the unit. He’s got the multiunit and the unit. Now he can find *any* number on that line.”

The episode concludes with the class collaboratively generating three strategies for locating 8, each of which involves a coordination between the action of splitting, displacing, and counting. The first strategy is to count two displacements of the white rod with a starting point at 6 and then the end point at 8 (fig. 24a). The second strategy, suggested by a student, uses a red rod to locate eight since the red rod equals

**Fig. 23.** The length of a green Cuisenaire rod can be split into the length of three white rods, and reciprocally, the length of each of three white rods can be aligned end to end to form the length of a single green rod.

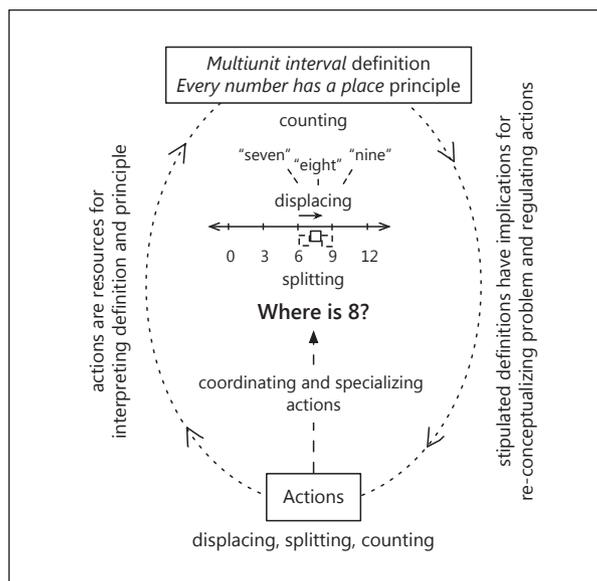


**Fig. 24.** Displacing and counting displacements of the white Cuisenaire rods and a red rod to locate 8.

two white rods (fig. 24b). Another student offers a third strategy suggesting a single counted displacement of a white rod moving in the other direction, from 9 to 8 (fig. 24c).

In sum, the *Correcting the Teacher* practice described in this episode creates opportunities for the class to make further sense of and extend the use of number line definitions, with prior definitions supporting the coordinated use of counting, splitting, and displacing actions. Figure 25 (adapted from fig. 10) displays the (simplified) dynamics of an interplay between definitions and actions supported by the teacher. In earlier lessons, actions of splitting, displacing, and counting previously afforded an interpretation of the definition and principle of *multiunit interval* and *every number has a place*. In turn, now the definition of *multiunit interval* and the principle of *every number has a place* have implications for the conceptualization of the problem; they give warrant to the assertion that 8 exists on the number line, contrary to the teacher's claim. They also provide an avenue to conceptualize an approach to locating 8 through a coordination and specialization of splitting, displacing, and counting actions. It is noteworthy that the use of splitting actions, constrained by the idea of a multiunit, foreshadows what will appear as students make an eventual transition to fractions in lesson 10, when unit intervals will be split into segments of equal length.

**Fig. 25.** An interplay between actions and definitions in locating 8 on a number line that is labeled in multiples of three.

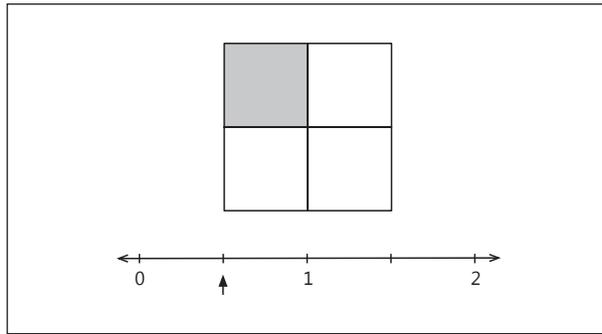


*Episode 5: Correcting the Teacher – Unit-Subunit Relations.* In lesson 10, the first lesson on fractions, Mr. K orchestrates a *Correcting the Teacher* practice that supports a transition from integers to fractions. The practice serves to carry forward the familiar definition of *unit interval*, but now used to serve a new function, one in which *unit interval* is constitutive of the definition of a *fraction*. For example, the *unit interval* now serves as a means to bound a distance on a number line, a distance that can be split into congruent parts in order to identify a denominator and numerator. This shift in function will have implications for new specializations and coordinations of actions that are elaborated in later lessons.

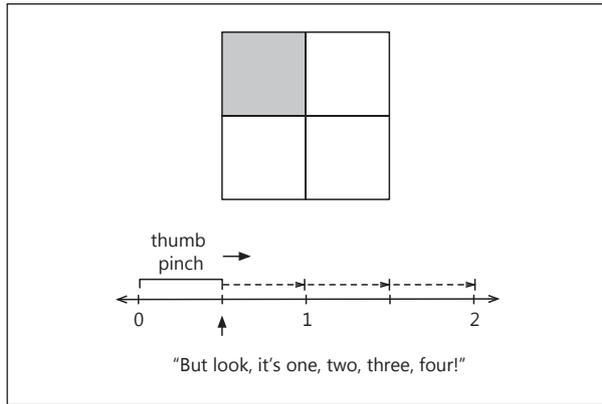
In the episode, the class is discussing a multiple-choice problem in which they must translate an area model representation of a fractional quantity (a square split into quarters with one grayed) into a number line representation. The sequence of *Correcting the Teacher* occurs as students are gathered around Mr. K at the overhead projector. The discussion has turned to option (a), which features an arrow pointing to an unlabeled tickmark as shown in figure 26 (the point for  $\frac{1}{2}$ ).

After establishing with the class that the shaded area of the rectangle split into four equal parts is  $\frac{1}{4}$  of the whole rectangle, Mr. K first asks, “what’s wrong with (a)?” Several students accurately reply in chorus that the arrow shows one half, not one fourth. Mr. K initiates *Correcting the Teacher* by advocating for option (a) as the correct answer. He justifies his position by pinching the interval between 0 and  $\frac{1}{2}$  between his index finger and thumb and displaces it four times to reach 2, imploring, “but look, it’s 1, 2, 3, 4” (fig. 27). He points out that the arrow is at the first of the four intervals. Over students’ objections, he repeats the displacement while counting more slowly; “there’s four equal parts, and this is just one of them.” The students continue to insist that option (a) shows  $\frac{1}{2}$ . Apparently unsatisfied with the students’ rebuttal, Mr. K demonstrates his justification yet again – this time even more emphatically –

**Fig. 26.** Opening discussion of option (a), an incorrect translation of  $\frac{1}{4}$  of a square to the indicated fractional point on a number line.



**Fig. 27.** Mr. K's demonstration that option (a) is correct in the *Correcting the Teacher* practice as he ignores the unit interval on the number line; he measures and displaces, and counts the interval distance asserting that the targeted point is  $\frac{1}{4}$ , not  $\frac{1}{2}$ .



repeating his counted translations that indicate that the arrow points to one fourth. Finally, Mr. K relents after hearing Lila's statement in which she asserts, "Because that is one piece and it has to be between one." Mr. K re-voices Lila's statement, incorporating the definition of *unit interval*: "Lila says it's gotta be inside a unit interval." He draws a bracket on the interval between 0 and 1 on the line. "That's where we gotta look; we gotta forget about that part (the interval between 1 and 2)." To conclude, Mr. K reasserts the importance of identifying the unit interval in representing fractions and then, with that logic, recognizes that the arrow in option (a) does indeed point to the position for  $\frac{1}{2}$ .

Noteworthy in this episode of *Correcting the Teacher* is a novel relation between definitions and actions. In particular, the idea of a splitting action is applied to the unit interval. These actions support students' sense-making of the *subunit* definition, formally introduced later in the lesson. In turn, the definition of *subunit* will come to act as a basis for regulating action coordinations that then become a resource for emerging definitions such as *denominator* (the number of subunits that constitute a unit), and *numerator* (the number of subunits from zero). Additionally, these actions are specialized to the number line in that they treat the unit interval as a constitutive part of a linear metric space. Thus, we again find a bootstrapping of actions and definitions, with actions used to make meaning of newly introduced definitions, and definitions, in turn, used as resources to regulate coordinations of

actions. In the next episode, which takes place during a later fractions lesson, we find additional ideas seeded by the interplay between action coordinations and definitions.

*Episode 6: Investigating Equivalent Fractions through Defining and Correcting the Teacher Practices.* Lesson 15 engaged the class with a definition for *equivalent fractions*, the idea that a single point on a number line can be expressed with a limitless number of expressions (e.g.,  $\frac{1}{2}$ ,  $\frac{2}{4}$ ,  $\frac{4}{8}$ , ...). In the episode that we identified, we found two instances of the *Correcting the Teacher* practice that are embedded within an extended *Defining* practice. The episode brings forward definitions elaborated in prior fractions lessons, including the *length of subunit* principle, *denominator*, and *numerator*. Over the course of the episode (and in each collective practice), we find that students' coordinated actions of splitting, displacing, and counting afford them the opportunity to develop a grounded conceptualization of equivalent fractions, a challenging mathematical idea.

For some students, *equivalent fractions* could present a contradiction with their prior knowledge. Indeed, canonical number lines have one and only one label assigned to tickmarks, and the *every number has a place* principle could be interpreted as indicating that every position on the number line is associated with only one expression. That stated, by lesson 15 students had significant conceptual resources to address the potential problem. Indeed, in lessons 12 and 13, students had explored improper fractions (e.g.,  $\frac{5}{4}$ ) and mixed numbers (e.g.,  $1\frac{1}{4}$ ), creating different representations for the same points on a number line through different coordinations of actions specialized for the number line; further, in lesson 14, they had explored the idea that whole numbers could also be represented as fractions on the number line (e.g., 1 could be represented as  $\frac{2}{2}$ ).

Three previously introduced terms provide students resources for conceptualizing equivalent fractions in lesson 15: the definitions of *numerator* and *denominator*, and the *length of subunit* principle (fig. 28). The *length of subunit* principle encapsulates the idea that, as the length of the subunit constituting a unit decreases, the number of subunits constituting the unit increases, leading to an invariant multiplicative relation for equivalent fractions. Though these ideas would seed the definition of *equivalent fractions*, students had yet to generate explicit knowledge that any point could be expressed with a limitless number of equivalent fractions. Neither had they developed well-grounded knowledge about the logic of equivalent expressions implied by the *length of subunit* principle and the definitions of *denominator* and *numerator*.

The lesson on equivalent fractions begins with students working independently on a number line problem. The problem requires students to write three fraction names for the same point (fig. 29). During the subsequent whole-class discussion of the problem, both the *Defining* practice and the *Correcting the Teacher* practice occurred while Mr. K worked with the class at the overhead projector with the problem projected on the overhead screen.

To generate three fraction names, the class first identified that the number line was not partitioned into equal intervals (fig. 29); Mr. K, at the overhead, split the interval between the midpoint and the point labeled "1" into two, creating four subunits using a coordination of splitting, displacing and counting actions. In this process, the denominator was then identified through a count of four subunits and the numerator

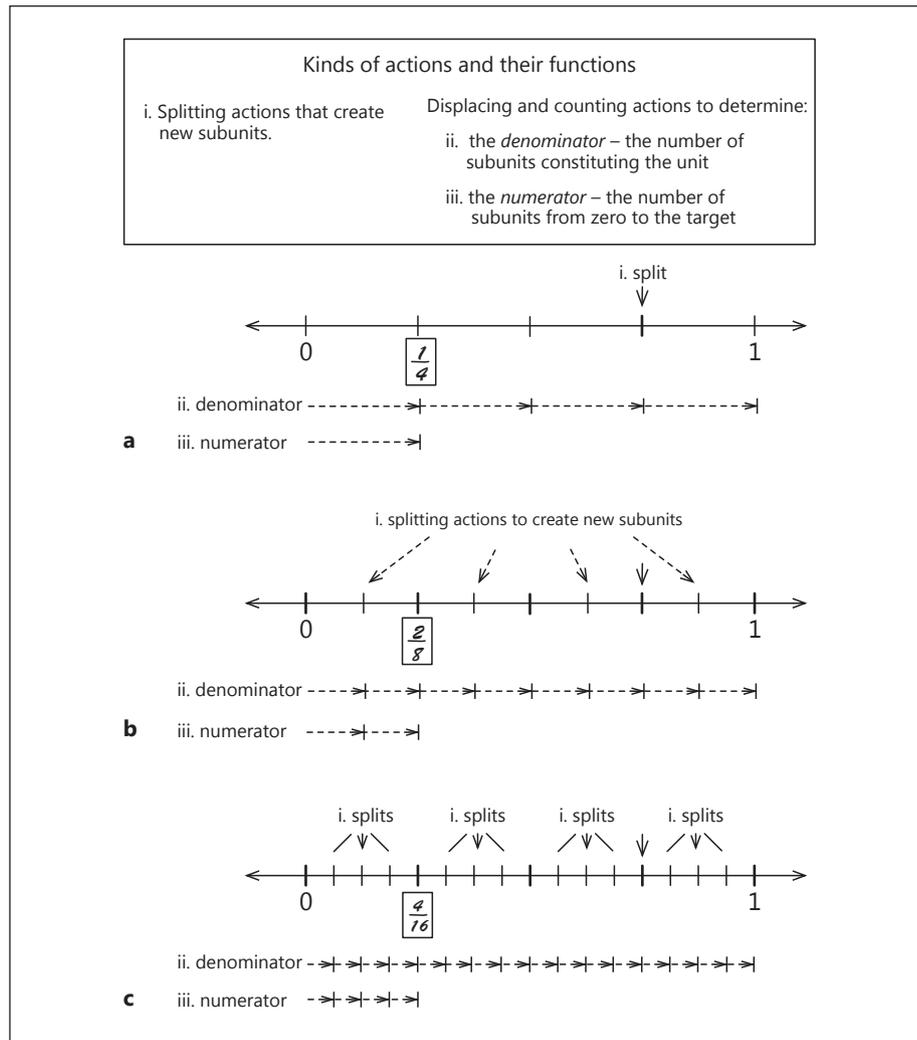
Length of the subunit	The more subunits in a unit the shorter the subunits are.	
Denominator	The number of subunits in a unit.	
Numerator	The number of subunits.	

**Fig. 28.** Principle and definitions posted in lesson 11: *length of subunit* principle, and definitions for *denominator* and *numerator*.

Write three fractions that belong in the box below. Make sure that each box has a different fraction!

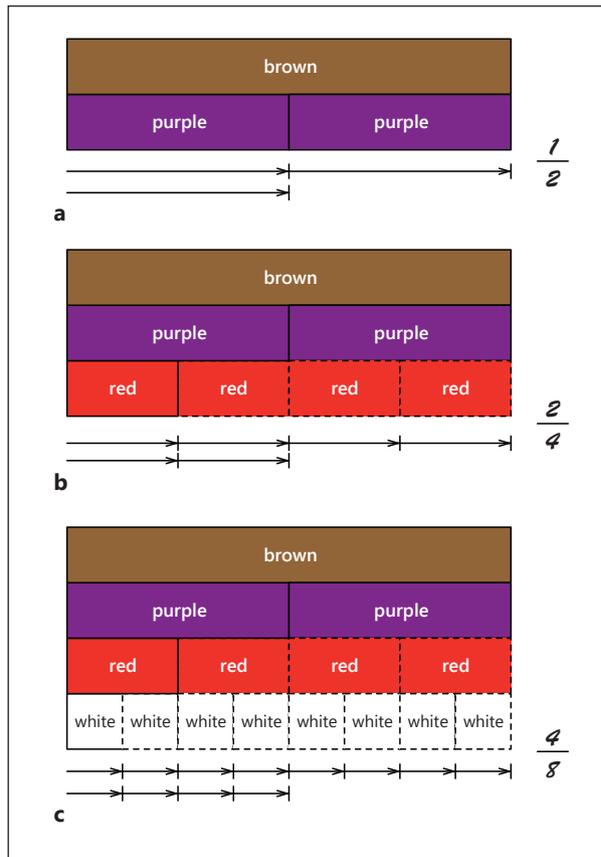
**Fig. 29.** An opening problem used in the lesson on equivalent fractions.

was identified through a count of one; the resulting representation is  $\frac{1}{4}$  (see fig. 30a). Subsequent coordinations of splitting, displacing and counting actions were used in the production of the representations of  $\frac{2}{8}$  and  $\frac{4}{16}$  (fig. 30b and c, respectively). However, prior to the generation of  $\frac{4}{16}$ , Mr. K initiated the *Correcting the Teacher* practice, claiming that there were no more fractions possible. Students object and the class produces the label of  $\frac{4}{16}$ . The resulting three labels –  $\frac{1}{4}$ ,  $\frac{2}{8}$ , and  $\frac{4}{16}$  – become the target of an ostensive definition. Mr. K states, “All of those [fractions] are the same; they are all equal. Who knows what we call them?” at which point the class responds in chorus, “equivalent fractions!”, having become familiar with the formal expression in a previous lesson (and perhaps in a prior classroom).



**Fig. 30.** The opening discussion: the production of three fraction labels for a point on the number line through coordinations of splitting, displacing, and counting actions.

After the production of an ostensive definition of *equivalent fractions* on the number line, Mr. K shifts the class's focus to Cuisenaire rods (fig. 31). He guides the class again through the production of Cuisenaire rod displays that would become the target for additional ostensive definitions, with the production of  $\frac{1}{2}$  of a rod's length,  $\frac{2}{4}$  of a rod's length, and  $\frac{4}{8}$  of a rod's length. We again find support for the coordination of splitting, displacing, and counting actions (now carried out with rods): Mr. K stipulates a unit length using the brown rod, and then "splits" the brown first into two subunits using purples to create a  $\frac{1}{2}$  split, then into four subunits using reds to iden-



**Fig. 31.** The construction of three displays of Cuisenaire rods to show the construction of expressions of  $\frac{1}{2}$  (a),  $\frac{2}{4}$  (b), and  $\frac{4}{8}$  (c).

tify the same point as  $\frac{2}{4}$ , and finally eight subunits using whites to identify the same point as  $\frac{4}{8}$ . For each set of splits, displacements, and counts, Mr. K writes the associated fraction expression on the whiteboard. Again, the representations become ostensive definitions of *equivalent fractions*:  $\frac{1}{2}$ ,  $\frac{2}{4}$ , and  $\frac{4}{8}$ .

In the culmination of the *Defining* practice, another instance of *Correcting the Teacher* emerges. Mr. K argues that  $\frac{2}{4}$  (written on the whiteboard) cannot be equal to  $\frac{1}{2}$  because 2 (the numerator in  $\frac{2}{4}$ ) is bigger than 1 (the numerator in  $\frac{1}{2}$ ). In objecting, a student presents a version of the *length of subunit* principle – as the subunits increase in length, the denominator gets smaller. As the *Defining* practice closes, Mr. K states and inscribes the formal definition on the poster: “Fractions that are the same place but with different subunits.” This formal definition integrates the splitting, displacing, and counting actions by means of which the class identified the same point with different fraction names. The definition sets the stage for subsequent lessons when students reason about multiplicative relations between numerators and denominators, ordering and comparing fractions on the line.

*Discussion: A Longitudinal Perspective on the Dynamics of Development –  
The Interplay between Definitions and Actions in the Collective Life of the  
Classroom Community*

When instruction in mathematics is successful, networks of historically elaborated ideas become generative for children, deeply integrated with their own developing thinking and problem solving. To understand how this process may be supported in classrooms over time, we drew upon the seminal contributions of Vygotsky and Piaget. Vygotsky's focus on the interplay between "scientific" and "everyday" concepts provided an entry into our treatment of relations between discipline-linked ideas that develop from the "top down" (Vygotsky's "scientific concepts") and ideas that develop from the "bottom up" (Vygotsky's "everyday concepts"). We made use of Vygotsky's argument that strong instruction entails a productive interplay between the development of children's everyday and scientific concepts. We also made use of Piaget's treatment of actions that afford the construction of logico-mathematical relations. For us, Piaget's treatment of actions, and the potential for their specialization and progressive coordination into logico-mathematical groups, served as a functional equivalent of Vygotsky's developing everyday concepts. They also provide a means of turning the number line into an artifact with logico-mathematical properties.

To support our longitudinal analysis of the interplay between processes of development and instruction, we selected the LMR curriculum unit. The choice served our purposes well, putting into clear relief core processes of interest. First, the curriculum provided a useful arena for longitudinal analysis. As we noted previously, in an experimental study contrasting LMR classrooms with controls, LMR classrooms achieved strong learning gains [Saxe et al., 2013a] that were reflected in Mr. K's classroom. Second, the curriculum supported an analysis of top-down developments since it privileged the introduction of discipline-linked mathematical definitions throughout the lessons that cover mathematical content often considered "hard to learn" and "hard to teach." Third, the curriculum also supported an analysis of bottom-up developments, since students were afforded the use of sensorimotor actions of displacing, counting, and splitting over their recurring engagement with number line problems.

Our inquiry resulted in a longitudinal case study of a single classroom community. Drawing on Saxe [2012], our analytic approach was to treat the classroom as a microculture with evolving interactional routines or "collective practices." We identified two collective practices that the teacher orchestrated in this classroom community: *Defining* and *Correcting the Teacher*, each of which became an arena for analysis. Our focus was on how the teacher orchestrated each of the practices to support a productive interplay between actions and the classroom community's stipulated definitions.

We found evidence of a productive interplay between actions and definitions in the case study classroom. Early in the lessons, actions of displacing, counting, and splitting provided an initial foothold for children to conceptualize and solve number line problems in generative ways. As definitions were stipulated, students' sense-making of definitions in terms of actions could support new action specializations and coordinations. This could in turn support a shift from the meaning making of stipulated mathematical definitions as objects to be understood into representational forms that could be drawn upon to regulate actions in potentially productive ways.

We also found that, across remarkably diverse mathematical topics, the teacher's orchestration of collective practices contributed to an evolving semiotic context of definition-action relations that supported coherence and connections across lessons. For example, in *Defining*, the teacher sometimes introduced new definitions as entailments of prior definitions, entailments grounded in actions. In *Correcting the Teacher*, the teacher contributed sustained use of familiar definitions in the context of new number line problems.

### Concluding Remark

We close by noting that our treatment engaged fundamental ideas related to cognitive development and instruction but that we elaborated these ideas in the context of a particular curriculum and a single classroom community. Nonetheless, we argue that this treatment can illuminate in productive ways the interplay between top-down and bottom-up cognitive developments in collective practices in other classrooms and in other subject matter domains. Regardless of the domain, teachers engage children with the equivalent of historically elaborated scientific concepts and procedures; at the same time, children come to the classroom with cognitive resources that they may be able to draw upon to make sense of these systems of concepts. The interplay between these top-down and bottom-up developmental processes takes form in microcultural worlds of classrooms that may support (to whatever degree) a productive interplay. Understanding these fundamental processes can shed light on longstanding questions about culture-cognition relations [Saxe, 2012], but at the level of microcultural worlds of classroom communities. It may also illuminate contemporary issues in education like the formation of learning trajectories [Clements & Sarama, 2009; Daro, Mosher, & Corcoran, 2011; Penuel, Confrey, Maloney, & Rupp, 2013; Simon, 1995] and participation in discipline-linked practices [Engle & Conant, 2002; Ford & Forman, 2006].

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